MONTHLY NOTICES

OF THE

ROYAL ASTRONOMICAL SOCIETY

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MEETING OF 1932 MAY 13

Dr. H. Knox-Shaw, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Wilfred Edward Candler, B.A., Trinity College, Cambridge ;
W. H. Darracott, 911 Jessie Avenue, Winnipeg, Canada;
Norman Fairclough, M.Sc., Huyton, 27 Alteryn Close, Newport, Mon. ;
James Chisholm Jenkinson, J.P., c/o Salvation Army, Uralla, N.S.W., Australia ;
Geoffrey Francis Kellaway, Coker Hill, West Coker, Yeovil, Somerset ;
Cecil Harris Leonard, B.Sc., Arwell, Pen-y-groes, N. Wales ;
Maurice Rupert Meehan, The High School, Marlborough, Wilts ; and
Mrs. Natalia S. Samoilowa-Jachontowa, Astronomical Institute, Leningrad, and 19 Karpovka, Ap. 41, Leningrad 22, U.S.S.R.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended :—

William Thomson Hay, 45 The Chase, Norbury, London, S.W. 16 (proposed by H. Knox-Shaw) ; and
J. M. Mohr, Physics Dept., University of Bratislava, Czechoslovakia (proposed by W. W. Heinrich).

One hundred and one presents were announced as having been received since the last ordinary meeting, including :—

Catania Observatory, Astrographic Catalogue, vol. 1, parts 5 and 6 ($+47°$) ;

Specola Vaticana, Astrographic Charts, zones $+55°$ and $+56°$ (10 charts);
R. T. Gould, "The Restoration of John Harrison's Third Timekeeper";
A. L. Loomis and W. A. Marrison, "Modern Developments in Precision Clocks";
Paris Observatory, "Catalogue de 784 étoiles de repère de la zone $+18°$ à $+24°$ (Paris) équinoxe 1925·0 observées à l'Observatoire National de Paris et à l'Observatoire Royal de Belgique."
Lantern slide of Nebula N.G.C. 1514 (presented by Lady Lubbock).

The President presented the Gold Medal of the Society to Dr. R. G. Aitken, who then delivered the George Darwin Lecture, taking as his subject, "What we know about Double Stars."

WHAT WE KNOW ABOUT DOUBLE STARS.

(George Darwin Lecture, delivered by Dr. R. G. Aitken, Assoc.R.A.S., on 1932 May 13.)

The history of double stars, regarded as physical systems, really began on 1802 July 1, when Sir William Herschel presented to the Royal Society the famous paper in the course of which he enumerated the "parts that enter into the construction of the heavens." In the first section, it will be recalled, he considered the "Insulated Stars"; in the second, "Binary Sidereal Systems or Double Stars," defining a binary or "real double star" as "the union of two stars, that are formed together in one system, by the laws of attraction."

In making this statement I am not unmindful of the fact that the first double star had been discovered as early as 1650, nor have I forgotten that Herschel himself began to search for pairs of stars to serve his plan for the determination of stellar parallax as early as 1779, and that Michell, in 1783, argued on the theory of probability that the majority of Herschel's early pairs were physical systems. Herschel's definition of a "real double star," it is important to note, rested not only upon theoretical arguments, but also upon a solid, though in the nature of the case, a rather small base of observational evidence. This evidence, the results of the actual measurement over a period of a score of years of the motions in the systems of Castor and of a few other real doubles, he presented to the Royal Society on 1803 June 9.

With these two papers Sir William Herschel laid the corner-stone of the edifice of double-star astronomy, and it seems appropriate, in addressing the Society of which he was the first president and on the occasion of the presentation to me of the Society's Gold Medal for work on double stars, to pass in review the progress that has been made, during the 130 years that have intervened, in raising the great structure which he founded; or, in other words, to call attention to what we have learned of double-star systems,

their orbital motions, their number, their distribution and origin, their relation to single-star systems—in brief, their function in the general economy of our stellar universe ; and to indicate, as well as I can, the most promising lines of attack upon the many problems they still present to us for investigation.

It is evident from this introductory statement that I have no novel theories to propound, no strikingly new generalisations to present, little to say, in fact, that is new. And I may add that, in my opinion, a general discourse like the present one is not the proper medium for the presentation of new theories ; these, I think, should be presented under conditions of time and place that will permit general discussion and free criticism.

Sir William Herschel laid the corner-stone, but the solid foundations for the structure we are considering were set by Wilhelm Struve and Sir John Herschel, the work of the latter at the Cape of Good Hope in the years 1834 to 1838 being as fundamental with respect to the double stars in the far southern heavens as was that of Struve, embodied in the *Mensuræ Micrometricæ*, for those observable from northern stations.

It is unnecessary to describe in any detail the work of the Herschels and Struves and of Dawes, Dembowski and their many other successors in later generations, for it is well known. It is sufficient for my purpose to remind you that, in the northern hemisphere, double-star observation was carried on with enthusiasm and success throughout the nineteenth century, nearly every astronomer who had a suitable telescope at command engaging in it to some extent, and many making it a major part of their working programmes. Search for new pairs, however, languished for a generation after the publication of Otto Struve's Pulkowa Catalogue, the attention of observers being concentrated mainly upon the measurement and remeasurement of the Σ and OΣ pairs.

As measures accumulated and gave evidence of relative motion of the components in numerous pairs, methods of deriving orbital elements were developed and many orbits were computed. A point of importance to note, in this connection, is that the chief objective of all this early work was to gain a better knowledge of the law of attraction that binds the components of a pair into one system. It could hardly be otherwise, for practically nothing was known of stellar distances (the *Mensuræ Micrometricæ* appeared a year before Bessel measured the parallax of 61 Cygni) and very little of the number, distribution and motions of the stars. It was natural and logical to assume that the law of attraction in these systems was identical with the law obtaining in our solar system, and while double-star measures even to-day are not accurate enough to prove that the law of inverse squares holds precisely, it sufficed long ago to show that it does hold within the limits of observational error. Our present-day interest in the motions in double stars has passed, therefore, to other problems for whose solution the measures of these motions provide data.

It should, however, be noted that Wilhelm Struve realised that double-star measurements would provide the means of determining what we now call the dynamic parallaxes of stars, and that in the Introduction to the

Mensuræ Micrometricæ he actually computed some parallaxes, on the basis of roughly computed orbits, with results in remarkably good agreement with values determined by direct methods in more recent years.

Interest in the discovery of double stars was revived when S. W. Burnham, an amateur astronomer, published a paper in the *Monthly Notices* of this Society in 1873, announcing the discovery of 81 new double stars with a 6-inch telescope set up at his home in Chicago. The new pairs were in all respects worthy of comparison with the Σ and OΣ pairs, and they proved beyond question that the field for such discoveries was by no means exhausted.

Burnham thus opened a new epoch in double-star astronomy, an epoch that has been most fruitful in discoveries and in the accumulation of measures. He, himself, was a most enthusiastic and tireless worker in this field, but always with the spirit of a "free lance," an amateur. He never worked systematically in searching for new pairs. It was his custom rather to identify and measure known pairs and then to examine stars in their neighbourhood that attracted his attention. All of his discoveries resulted from such random search.

His success stimulated others to search, and hundreds of additional pairs were discovered by Hough, by See and others, with the result that in 1896 Burnham estimated the number of known pairs within 120° of the North Pole to be nearly 11,000, half of which, perhaps, would meet the more exacting requirements of our present-day definitions of a double star.

But while these discoveries demonstrated the incompleteness of the earlier surveys by the Struves, they afforded no basis for the investigation of statistical questions relating to the number and distribution of double stars because they were all the result of random search. It was to provide data for such statistical studies that I began my survey at the Lick Observatory in 1899 April. I decided to chart all the stars as bright as 9·0 in the Argelander and Schönfeld Durchmusterungen and to examine every one of them with the telescopes available at Mount Hamilton. You are aware that Professor Hussey joined me in this project a few months after I had initiated it and that we worked in close co-operation until he left the Lick Observatory in 1905. After that, I completed the work, which required far more time than I had originally anticipated.

Such a systematic survey has not only the advantage of providing data for statistical studies, it also adds markedly to the number of discoveries of new pairs. A considerable percentage of the pairs of my own discovery with the 36-inch refractor, for example, are quite within the range of a good 6-inch telescope. They had not been discovered earlier simply because they had not been examined.

The immediate, and probably, in the long run, the most important, result from the Lick Observatory survey was the discovery of 4437 new pairs—1329 by Hussey, the others by me—nearly all of which were of 9·0 magnitude or brighter and under 5″·0 in angular separation. I have no hesitation in affirming that, with but few exceptions, these new pairs are physical systems —" true double stars."

To bring the story of double-star discovery in the northern heavens to date, brief reference must be made to the many short lists of new pairs that have been published within the last twenty-five years and to the notable series of discoveries by Robert Jonckheere and by T. E. H. Espin (assisted in recent years by W. Milburn). Espin's work is specially to be noted since his programme includes the examination of all stars to the limit of 9·5 magnitude (B.D.). It is true that the great majority of pairs in his list, as in Jonckheere's, are comparatively faint and wide, and that it will be long before repeated measurements of them add greatly to our list of systems in known orbital motion, but because his search is a systematic one, it should provide a useful supplement to the Lick Observatory survey.

Double-star discovery and observation in the southern hemisphere were carried on in but a desultory manner after the time of Sir John Herschel until 1896, when R. T. A. Innes was called to the Royal Observatory at the Cape of Good Hope. Already interested in double-star work, he found opportunity here, and later at the Union Observatory, to carry on his researches under more favourable conditions. Both at the Cape and at Johannesburg he was most active in measuring known pairs and in discovering new ones, and at an early date realized the desirability of extending to the South Pole a survey of the sky on the lines of the one in progress at the Lick Observatory. I think I am betraying no confidence in saying that when he first secured the promise of a large refractor for the Union Observatory he invited me to come to Johannesburg, when it should be ready, to co-operate in such a programme. But that was more than twenty years ago.

Unforeseen difficulties delayed the completion of the telescope and it was not erected until 1925. By that time a most fortunate alliance had been made between the observatories at Leiden and Johannesburg, and the survey was actually inaugurated under the capable direction of Dr. W. H. van den Bos. Though still incomplete, it has already borne the richest fruit. Innes's own discoveries (including those of earlier date) number 1613, Finsen's 300, while those by van den Bos exceed 2000. These are all close pairs, comparable in every respect with those discovered at the Lick Observatory. What is equally important is that quite as much time is given to the remeasurement of known pairs, especially of pairs showing orbital motion, as to the search for new ones.

Another important programme of double-star work in the southern hemisphere is the one initiated by Professor Hussey. His interest in southern doubles was shown by his work at La Plata in the years 1911–1914, in the course of which he discovered 221 new pairs and made a long series of measures of known pairs. With the financial support of his classmate and friend, Robert P. Lamont, he later built a 24-inch telescope and organised an expedition to Bloemfontein, S.A. You will all recall the tragedy of his death here in London, when, all preparations at last completed, he was on his way to begin his work at that station, and you will rejoice with me that work such as he had planned in the discovery and measurement of double stars is now in progress there under the direction of Mr. R. A. Rossiter. It is being carried

out in co-operation, in a general way, with the work at the Union Observatory, and while no results have yet been published, it is known that several thousand double stars have been discovered and measured.

To put the wealth of double-star material that has been gathered by the zeal of so many enthusiastic observers in all parts of the world at the ready command of astronomers, including the double-star observers themselves, it is essential that general catalogues be compiled from time to time. Passing over earlier compilations, it will suffice to note that the two most comprehensive and generally useful ones in recent years have been Burnham's *General Catalogue of Double Stars within* 121° *of the North Pole*, published in 1906, and the *Southern Double Star Catalogue*, $-19°$ *to* $-90°$, published by Innes, with the collaboration of Dawson and van den Bos, in 1927. Each of these catalogues was designed to include all pairs, within its declination limits, that had been published and measured as double stars, and each has proved invaluable. Innes's catalogue, however, was issued frankly as a provisional one, to be superseded by a more definitive edition as soon as the survey of the southern sky should be completed, and so great has been the activity of observers in the northern hemisphere in the years since 1906 that Burnham's catalogue has long needed revision. Jonckheere's supplement, issued in 1918, was helpful, but rather as a list of later discoveries than as a complete history, even of the pairs it listed.

As early as 1914, my good friend the late Eric Doolittle began the compilation of a card catalogue, to be published in due time as an extension to Burnham's work, and after his untimely death in 1921 I took over this material, in fulfilment of a promise made to him two years earlier, and brought it to completion. Unexpected difficulties prolonged the work of compilation, and it has required more time than I had anticipated to see the work through the press, so that the new catalogue is only now being distributed. It is a pleasure, indeed, to be able to-day to present the first copy to the Society.

The new catalogue does not duplicate the data given by Burnham, and is in that sense a supplement to his work. In all other respects it is an independent piece of work and differs decidedly from Burnham's in that it sets definite limits, based upon angular separation and magnitude, within which a pair must fall to be included in the catalogue. True, these limits were drawn generously, the curve adopted being represented by the formula, $\log \rho = 2{\cdot}8 - 0{\cdot}2$ m., which gives an angular separation of 10″ for a pair whose combined magnitude is 9·0. Even this limit was not too rigorously applied, since it was thought wiser to err by inclusion rather than by exclusion ; and yet it developed that nearly one-third of the 13,665 pairs in Burnham's Catalogue were excluded. It is, therefore, eloquent testimony to the activity of double-star discoverers in the years since 1906 that the new catalogue has 17,180 entries. When we add to this number the thousands of similar pairs in the zones south of $-30°$ declination that have already been discovered and measured, we find that we have hundreds of thousands of measures of not less than 27,000 visual double stars that will come within the limits set in the New General Catalogue.

Has it been worth while to make this truly prodigious collection of material ? And is it worth while to continue to add to it in future years ? Obviously, that depends upon what we have learned from it and what more we may reasonably hope to learn.

Let us consider first the number and the distribution of the visual double stars in our stellar system.* Since we are interested only in the "true double stars" we cannot base our conclusions upon a collection of material, like the New General Catalogue, that will certainly include a large number of purely optical pairs. It will be safer to consider only the pairs falling within the narrower limits set by the formula, $\log \rho = 2{\cdot}6 - 0{\cdot}2$ m., which gives the angular separation 5″·0 for a pair of 9·0 magnitude, for we may reasonably assume that any optical pairs still included will be offset by binary pairs that fall outside these limits.

By a single count I find that 12,708 of the double stars in the new catalogue fall within the limits just defined. These are given in Table I, with the arguments, magnitude and angular separation. Another count might give a slightly different distribution, for the magnitudes fainter than 8·0 are often in doubt and it is not always easy to determine the angular distance that should be adopted, but the numbers set down may be regarded as approximately correct.

TABLE I

Sep'n / Mag.	< 0″·51	0″·51–1″·00	1″·01–2″·00	2″·01–5″·00	5″·01 to Curve	Total
< 6·00	66	69	63	156	317	671
6·01–7·00	136	98	134	214	302	884
7·01–8·00	305	244	346	522	341	1758
8·01–9·00	879	793	1035	1564	363	4634
9·01 +	371	512	1151	2727	...	4761
Total	1757	1716	2729	5183	1323	12708
Per cent.	13·8	13·5	21·5	40·8	10·4	100·0
Omitting Line 9·01 +						
Total	1386	1204	1578	2456	1323	7947
Per cent.	17·4	15·2	19·9	30·9	16·16	100·0

The table is obviously incomplete for pairs fainter than 9·0 magnitude, and particularly for the closer pairs in this category. The reason is, of course, that no systematic examination of these fainter stars has yet been made under satisfactory observing conditions. It is not even entirely complete for the brighter pairs, since the Lick survey did not extend as far south as −30° and the southern survey has not fully covered the area between that parallel and the bounding parallels of my survey. The progression of the numbers in the first four lines, however, is reassuring, and I think warrants the following conclusion.

* My argument must rest upon the data for stars north of −30° declination, since the survey of the far southern sky is still incomplete.

One star in every eighteen, among those as bright as 9·0 *magnitude, is a close double star within the resolving power of modern telescopes.* This conclusion rests upon the tabulated total, 7947 pairs, and upon the most careful estimate I can make of the number of stars of corresponding magnitude actually examined. It is supported by my earlier investigation, based upon the 5400 pairs as bright as 9·0 magnitude in the northern half of the sky that fall within the limits set by my "working definition" of a double star, and also by van den Bos's preliminary result from certain southern zones. Van den Bos, in fact, finds an even higher ratio—1 to 16 or 17.

Nearly one-third of the pairs in the first four lines of the table have angular distances under 1″·00, more than one-half of them distances under 2″·00, whereas only one-eighth exceed 5″·00. This again confirms my earlier finding that there is a real increase in the number of visual double stars as the angular distance grows smaller. I am concerned in this address only with the visual binaries, or double stars, but here, as in part of what follows, it will be pertinent to appeal to the data afforded by the spectroscopic binary systems in support of my argument. Among the brighter stars it has been found that at least one in four, possibly one in three, on the average, is a spectroscopic binary. Although it does not yet appear that this percentage is maintained among fainter stars, this may be due, in large part, to incompleteness of the observational data. The spectroscopic binaries, on the average, have orbital dimensions far smaller than those of the visual binaries in the first column of Table I, and their numbers thus strengthen the conclusions I have just stated.

When the 12,708 pairs of Table I are plotted on equal area projection charts together with the central line and the poles of the Milky Way, the concentration toward the plane of the Milky Way is clearly in evidence, but since more than one-third of the pairs are fainter than 9·0 we cannot say definitely that it is greater than that of all stars to the corresponding limit of magnitude.

A curious feature in the distribution of these double stars is brought to light when they are tabulated separately for the four quadrants of right ascension, as in Table II. I refer to the relatively small number of close pairs in the fourth quadrant.

TABLE II

Sep'n / Quad.	<0″·51	0″·51–1″·00	1″·01–2″·00	2″·01–5″·00	5″·01 to Curve	Total
I	490	482	647	1318	377	3304
II	417	409	688	1154	298	2966
III	290	298	471	742	220	2021
IV	560	527	923	1969	438	4417
			Per Cent.			
I	14·8	14·6	19·6	39·9	11·1	100·0
II	14·1	13·8	23·2	38·9	10·0	100·0
III	14·4	14·7	23·3	36·7	10·9	100·0
IV	12·7	11·9	20·9	44·6	9·9	100·0

It is not an observational effect, for the fourth quadrant is precisely the one that is most favourably placed for observation at Mount Hamilton and at other northern stations where double-star work has been carried on actively. If the close pairs were there, they should have been discovered. It would be rash, however, to read into this table or into the diagrams showing the apparent distribution of these pairs anything more than general hints at the true distribution of double stars in space, for too great uncertainty still attaches to the data. Here, as often elsewhere, it is well to keep Sir Thomas More's quaintly worded warning in mind. "Of all this pointe," he writes, "there is no certaintie, and whoso divineth uppon conjecture may as wel shoote too far as too short." After the survey of the southern sky has been completed, an investigation based upon accurate photometric magnitudes and upon angular distances reduced to a common epoch and including data for the whole celestial sphere, may give results of significance.

But if double-star discovery, in itself, has comparatively little value even when conducted on a systematic plan, it is none the less essential since it is the necessary pre-requisite to double-star observation, as Sir John Herschel pointed out many years ago; and when we pass to the consideration of results based upon the data accumulated by the patient and painstaking measurement of double stars year after year and decade after decade, we come upon firmer ground. Even here progress is necessarily slow, however, for these systems are drawn upon a gigantic scale. The apparent orbital motion in the vast majority of pairs becomes appreciable only after they have been kept under observation for many decades, if not centuries, and we must remember that most of the systems of smaller dimensions have only recently been discovered.

An examination of the data in the New General Catalogue leads me to estimate that we have actual observational evidence of orbital motion in about one-fourth of the 12,708 pairs that fall within the limits set by the curve, $\log \rho = 2{\cdot}6 - 0{\cdot}2$ m. In at least half of these cases, however, the observed change is but little greater than the error of measure, and for only 1495 or approximately 12 per cent. of the whole do I regard the evidence as really convincing. When these pairs are grouped according to spectral class

TABLE III

Spec. / Sep'n	B	A	F	G	K	M	Un-known	Total	Per Cent.
$< 0''{\cdot}51$	11	161	107	211	79	2	112	683	45·7
$0''{\cdot}51$–$1''{\cdot}00$	7	62	46	100	38	2	31	286	19·1
$1''{\cdot}01$–$2''{\cdot}00$	5	48	42	92	45	3	16	251	16·8
$2''{\cdot}01$–$5''{\cdot}00$	3	31	26	77	54	6	9	206	13·8
$5''{\cdot}01$–curve	0	17	8	17	17	8	2	69	4·6
Total	26	319	229	497	233	21	170	1495	...
Per cent.	1·7	21·4	15·3	33·2	15·6	1·4	11·4	...	100·0

and angular separation, as in Table III, we find that one-third, or, omitting those of unknown spectral class, three-eighths, are of spectral class G and

46 per cent. or nearly one-half have angular distances under 0″·51. There are very few pairs with angular distances exceeding 5″·00 and fewer still that belong to classes B and M. This general distribution in spectral class is now well established, but so far as the numbers in class G are concerned it is doubtless largely a selective effect. The numbers in classes B and M may be more significant. It will be recalled that pairs of spectral classes B and A predominate among spectroscopic binaries, more than half of those in Moore's Third Catalogue having spectra earlier than A5, whereas spectroscopic binaries of class M are as rare as visual binaries of that class. The small number of visual binaries of class B, then, may signify that binary systems of this class are, in general, too close to be separated with the telescope. Binaries of class M, on the other hand, may actually be exceptional.

Orbits can as yet be computed for but a very small fraction of these 1495 pairs : indeed, the last and most complete catalogue of fairly satisfactory orbits, that by van den Bos, listed only 117 pairs, though the systems south of $-30°$ were also included. Orbits of a few other pairs have since been added, but several of those in the catalogue have been proved to be unreliable, so that the total remains almost unchanged. Our interest in these orbital motions, as I have remarked, no longer centres upon the testimony they bear to the law of attraction controlling them, but rather upon the light they can throw upon the problem of the origin of binary systems and even more upon the information they can give us as to stellar masses and other physical attributes. Slight as our stock of material is, it is still sufficient to give us some very definite facts in the one case, and to show us what appear to be promising approaches to a solution in the other.

I do not wish to imply that we have lost all interest in the gravitational problems these systems present. When but two bodies are concerned, we are indeed satisfied that the true orbit is a Keplerian ellipse, or at least that motion in such an ellipse represents the observations within the error of measure. But there are numerous systems of three or of four bodies ; in fact, counts of two long sections in the new catalogue confirm my earlier estimate that these multiple systems constitute from 4 to 5 per cent. of the whole. In general, the angular distance to the third body, or, alternatively, to the second couple, in these visual multiple systems, is large in comparison with the angular separation of the main pair—a fact of interest in the theory of their evolution. Not infrequently we find systems, like κ Pegasi or 13 Ceti, in which one of the components of the visual pair is also the visible member of a very close spectroscopic binary ; or systems, like Castor, in which each component of the visual system has a companion revealed only by the spectroscope. In general, the micrometric measures of the visual pair, in such cases, are not sensibly affected by the existence of the unseen components. But there are still other systems, in which the unseen star revolves with one of the visible components in an orbit of longer period so disposed as to occasion marked periodic variations, or perturbations in the motions in the visual system. The problems these systems present are of fascinating interest. For ζ Cancri and ξ Ursæ Majoris, classic examples of this type, the solutions reached are reasonably good : in

the case of 70 Ophiuchi, to name but one of a number, the problem is more baffling and no satisfactory solution has yet been made, though a recent investigation at the Lick Observatory by Dr. Louis Berman, in which spectrographic observations have been used to supplement the visual and photographic measures, is illuminating.

A sharp distinction is to be drawn, I think, between systems composed of a binary pair and a third, relatively distant, companion, or of two binary pairs, visual or spectroscopic, separated by a distance many times greater than the major axis of the orbit of either pair, and groups of three or four or more stars, like the Trapezium in the Orion nebula, which are really star clusters. The dynamics, and doubtless the origin, of these groups are very different from those of the ternary and quadruple systems we have been discussing.

It is well known that double-star systems afford us the only direct observational evidence of stellar mass, and as rapidly as good orbits and good parallax values for different systems have become available, mass computations have been made. Though they show nothing essentially new, I have made such computations for all of the reliable available orbits of the visual double stars for which values of the absolute parallaxes are given in Schlesinger's *Catalogue of Bright Stars*.

The results, including also the mass of Kreuger 60 which is not in Schlesinger's Catalogue, are given in Table IV.

Table IV

Spectral Class	Number	$(m+m_1)$
B	1	10·65
A	15	5·21
F	12	2·56
G	34	2·42
K	11	2·15
M	2	0·64

The masses quoted in this table are the masses of the systems, not of the individual components. Only in those cases in which the motion of each component relatively to the centre of mass of the system has been determined by measures from external stars, can we compute the mass ratio of the two components and thus the mass of each of them in terms of the Sun's mass.

In spectroscopic binary systems, on the other hand, we can compute the mass ratio of the components in all cases where two spectra appear upon the plates (approximately one-sixth of those for which orbits have been published), but not the masses themselves, for $\sin^3 i$ is a factor in the mass function, and the inclination cannot be determined from measures of radial velocity. It remains unknown except in the case of an eclipsing binary. By assuming an average value for $\sin^3 i$, however, we can derive the average values for groups of systems, and when we take these groups according to spectral

class, we find in the average masses the same apparent dependence upon spectral type that is exhibited in Table IV. This, it is now recognized, is really a relation between absolute magnitude and mass, and when the masses are plotted against the absolute magnitudes for all systems in which both quantities are well determined, the points will all fall upon, or very nearly upon, a smooth curve. Eddington's investigations provide the physical explanation of the relation, and his mass-luminosity curve based upon purely theoretical considerations, but with what we may call its zero point fixed by the well-determined mass and absolute magnitude of the brighter component of Capella, is in remarkably close agreement with the one derived directly from the observational data.

Now we can reverse the process, and instead of deriving the masses from the orbit elements and parallaxes of double stars, compute the dynamic parallaxes from the elements and from masses assumed from the position of the star on the mass-luminosity curve. Indeed, we do not even need the orbit elements. If the observed arc suffices to define the relative accelerations of the two components at a given epoch, it has been shown by Hertzsprung and independently by Russell that a value of the dynamic parallax can be computed, and it appears, upon actually putting the question to the test, that these parallaxes, on the average, are affected by a statistical probable error of less than 25 per cent. The longest series of actual computations of dynamic parallaxes, so far published, are the one by Jackson and Furner and the later one by Russell and Miss Moore. The method, of course, cannot take account of what I may term the star's individuality. Not all stars of the same mass have precisely the same absolute magnitude, and to the extent by which a particular system deviates from the mean in this respect, its dynamic parallax will also be in error. There are still hundreds of double stars for which no dynamic parallax has been computed, though the requisite data are available, and the measures of the close pairs of recent discovery ought within a few years to add the data for still other hundreds.

Much attention has been given to tracing possible correlations between the different orbit elements of binary systems and between certain of these elements and fundamental planes of reference in our stellar system. Quite recently, for example, M. D. Barbier published a paper in the *Journal des Observateurs*, in which he shows that when the orbits of the spectroscopic and visual binary stars are grouped in zones according to their galactic latitude and longitude, the eccentricities and periods increase with increasing galactic latitude, and are larger in the hemisphere with galactic longitudes $180° - 270° - 360°$ than in the one with galactic longitudes $0° - 90° - 180°$. What significance such a correlation may have, if it is definitely established, does not appear ; and this remark holds for all other correlations for which more or less evidence has been adduced, with the single exception of the correlation between eccentricity and period. Not only is this established beyond reasonable question, but it is also probable that it has a physical basis and is in some way related to the origin and evolution of binary-star systems. Table V, drawn up by Moore and myself on the basis of 226

orbits of spectroscopic binary stars and 103 orbits of visual double stars, brings out this correlation clearly :

TABLE V

Number	Average Period	Average e
	Spectroscopic Binaries	
	Days	
83	2·7	0·05
39	7·6	0·16
29	14·1	0·22
23	30·6	0·35
21	102·5	0·30
31	1177 (=3·2 years)	0·31
	Visual Binaries	
	Years	
14	16·8	0·43
24	37·1	0·40
24	73·0	0·53
23	138·0	0·57
18	274·3	0·62

Unquestionably the eccentricity, on the average, increases with the period ; the spectroscopic orbits of shortest period are very nearly circular, the visual orbits of longest period are greatly elongated ellipses.

Russell has extended the investigation of this relationship, statistically, to the wider pairs for which orbits are not yet available, and finds the average eccentricity of 500 pairs, of average period roughly estimated at 2000 years, to be 0·61, and "of nearly 800 more with average periods of perhaps 5000 years," to be 0·76. These values are, of course, far more tentative than those given in the table, but they may be quoted as corroborative evidence.

Probably every observer of the closer visual double stars has shared my experience in finding that when the two components are of approximately equal magnitude they are indistinguishable in colour. I have noted this particularly in the hundreds of very close pairs—those with angular distances not exceeding 0″·3—which I have measured. In at least nine cases out of every ten the difference in magnitude is less than $0^m·5$—generally much less—and there is no appreciable difference in colour. In all probability the two components in these pairs are also of the same spectral class, though the only observational evidence on this point, in addition to the colour, is the negative testimony given by the fact that they do not show a composite spectrum.

When the pairs are wider the spectra of the two components may be determined separately, and several investigations of this character have been made, the most thorough study being that by Dr. F. C. Leonard. He finds that when the two components are equal or nearly equal in brightness, their spectra are almost always very nearly alike. As the difference in magnitude increases, so does the difference in spectral class, the general rule being that the fainter component is of the later type—redder. There

are pairs, however, in which the contrary relation appears, the fainter component being of earlier type—bluer—than the primary. The distinction between the two classes of systems appears to be one of absolute magnitude. In systems of the first class both components are dwarf stars, or if the primary is of class A or B it, like its companion, belongs to the main stellar sequence. In systems of the second class, the primary is always a giant star, with absolute magnitude brighter than +2. This is the case also in the few systems in which the two components, though of greatly different magnitudes, are of approximately the same spectral class. Doubtless, this correlation, like the one between orbital period and eccentricity, finds its explanation in the evolutionary history of the binary stars.

What, then, do we know of this history? The sad fact is that as yet we really know very little. The successive configurations that will be assumed by a rotating fluid mass acted upon by no external force have formed the subject of extensive mathematical researches by many distinguished investigators, among them Sir George Darwin and Sir James Jeans. The net result is to show that its ultimate fate *may* be fission into two bodies, not greatly unequal, revolving nearly in contact about their common centre of mass. Granting that this occurs, tidal forces will increase the period, the semi-major axis and the eccentricity of the orbit of each component within certain limits, and it is not impossible that, as Jeans thinks, the short-period spectroscopic binaries are the visible manifestations of such evolutionary processes. But even when chance encounters with other stars are invoked and the assumption is made that each component is slowly losing mass through radiation, these processes cannot result in systems with the gigantic dimensions and high eccentricities that characterise the visual binary stars of long period.

On the other hand, it is quite as difficult to explain the very short-period spectroscopic binaries on the assumption that they developed from the long-period visual systems. If the latter originated in condensations about separate nuclei in a primal nebular mass, the orbital motion of the components in such a resisting medium (even if its density be very low) would indeed tend to shorten the periods, lessen the orbital dimensions and decrease the eccentricities. Whether the masses would increase or decrease secularly would depend upon a number of conditions not easily defined. But it would require the most favourable assumptions in all particulars, not to mention almost unlimited time, to evolve a spectroscopic binary system of one-day period from a visual binary of even moderately long period.* Finally, we must also account for ternary and quadruple systems and for the combination of very long and very short periods in systems like that of Castor.

Admitting all of these as yet unsurmounted difficulties, we are still led to the conclusion that all binary systems had a common origin, for there is

* After this address was written, the March number of the *Astrophysical Journal* reached me. In his article on "The Evolution of Binary Stars," William Markowitz presents arguments favouring an evolutionary development from long-period to short-period pairs. I must say frankly, however, that I do not think he has established his thesis.

no definite evidence of discontinuity in the entire series from the closest spectroscopic binary to the longest-period double star.

So far, then, the observation of double stars has given us some knowledge of the motions in stellar systems, and of the laws governing them ; has revealed the fact that a system of two stars of co-ordinate rank is one of the standard products of stellar evolution, and that systems of three or even of four stars are by no means unusual phenomena. It has afforded some indications of the distribution of these binary systems among stars in general ; has put into our hands some clues which, if followed up, may lead us to a knowledge of their origin and development ; and, most important of all, perhaps, has given us, in conjunction with parallax determinations (of which more are sadly needed), fairly accurate mass values for a number of systems and of their individual components. This final item alone fully repays us for all the time and labour that has been devoted to the study of double stars, and no further argument is required to convince us of the desirability of continuing our observations.

It does not follow, however, that we should confine ourselves simply to the measurement of the motions within the systems, as we have so largely done in the past, or that even in these measures we should continue to employ only the methods and instruments that we have been using ever since the days of the Herschels and the elder Struve. We still rely mainly upon the filar micrometer for our measures of angles and distances, and must for long continue to depend mainly upon visual measures with some form of micrometer for measures of the closer double stars, which, on the average, are the ones that will most quickly give us the data we are eager to have. Quite possibly Hargreaves's comparison-image micrometer may prove to give more accurate measures, at least of angular distances, than the usual filar micrometer, and it is not too much to hope that the interferometer, after its technique has been more fully developed, and the sources of error to which measurements with it are subject have been traced out and their effects removed, may give us the relative positions of the components in the closest pairs more accurately than do any of our present micrometers. Certainly it is important that experiments along these lines continue, and experiments also in the use of the reversing prism in conjunction with the filar micrometer. But for pairs whose images do not overlap upon the photographic plate it is now evident that we can secure more accurate measures by photography than by any visual methods.

Photographic measures of double stars are not a new thing. The first photograph of a double star was taken by G. P. Bond as early as 1857, and in more recent years considerable series of measures have been made at a number of observatories in many different countries. At first serious difficulties were encountered and the measures were found to be subject to systematic errors that counterbalanced their greater freedom from accidental errors of measurement. But more recent investigators, among whom Hertzsprung has been specially prominent, have succeeded in devising methods that have largely eliminated the systematic errors, and further improvement as the outcome of continued well-planned experiments is

certain. It is already possible to secure photographs with our larger telescopes of pairs whose components are not greatly unequal in brightness, even if the angular distance is considerably less than 1″·0, and photographic measures of such pairs with slightly larger angular separation are more accurate and consistent than measures made visually. Further, measurable photographs can already be secured of wider pairs whose components differ in brightness by four or more magnitudes, even without the use of devices to reduce the brightness of the primary. Sirius and its companion, for example, with a light ratio of 10,000 to 1, and an angular separation of less than $9\frac{1}{2}''$, have been successfully photographed with the Crossley reflector. It is to well-planned photographic observation that we must look in the future for the accumulation of the accurate measures of motion in all systems to which the photographic method can be applied. More and more the visual observer should confine his attention to pairs beyond the present reach of the photographic process, and as he does so he will become ever more impressed with the fact that good atmospheric conditions are the prime essential for good measures.

And we must not limit ourselves to measures of the relative motions in double-star systems; we have reached a stage in our researches, I think, where it is fully as important to have an accurate knowledge of such attributes as the photometric and photographic magnitudes and the spectra of the individual components of a system as of their relative motions and positions. Such researches as the one by Leonard, already referred to, on the spectra of the components of double stars, and the one that Kuiper, under Hertzsprung's general direction, is carrying out on their relative magnitudes, are most highly to be commended.

When the survey of the southern sky that is now in progress shall have been completed, further double-star *discoveries* will become relatively unimportant. We shall not so much need more systems to study, as more and more accurate observations of those already known. To reach adequate solutions of the problems we already have had outlined for us in our double-star researches we require an accumulation of highly accurate measures of the motions, the magnitudes, the spectra, and the parallaxes of double star systems, so great that our present accumulation will appear, by comparison, insignificant.

1932 *March* 2.

THE ANALYSIS OF STELLAR STRUCTURE. II.

E. A. Milne. (Plate 10.)

The following paper presents a continuation of the ideas and investigations contained in "The Analysis of Stellar Structure," *M.N.*, **91**, 1, 1930.* It first develops somewhat further the fundamental principles, outlines the programme of investigations which is suggested by them, and indicates the

* Referred to as A.S.S., I.

significance of the results to be expected from the execution of the programme. The programme is then carried out in full for one particular system. Lastly, the results are utilised to give insight into the explanation of certain observed characteristics of the stars.

If in this paper I ignore the many criticisms of the former paper which have appeared during the interval, it is not for want of studying them or for lack of respect for their authors. It is rather that the essential point of my formulation of the problem has been somewhat misunderstood. I called the paper "The Analysis of Stellar Structure" designedly; it was an approach to the subject originating solely in an endeavour to see what could be inferred from observed data. My object was to determine on rational grounds the nature of configurations possessing the observed mass M and observed luminosity L of a star, and to compare the results (by calculating the external radii of the configurations) with the observed star. This is a different procedure from making hypotheses as to the nature of observed stars. I do not put forward a theory of stellar structure—I put forward a set of principles and a method. I investigate the different ways in which an aggregate of matter of mass M can be put together so as to be in a steady state when endowed with sources of energy totalling to L. Given sufficient thermodynamic information about the phases which matter is capable of assuming, this is a large-scale formal problem whose solution is perfectly accessible, whose solution, moreover, is independent of hypotheses about actual stars, of the mechanism of the generation of stellar energy or of any theory of stellar evolution.

My work did not of course originate in the use of singularity-possessing solutions of Emden's equation. The use of these solutions was simply the inevitable way of embodying the problem in mathematics—the problem, namely, of ascertaining the possible states of a mass M, observed to possess luminosity L.

In the following paper I use the word " phase " in its thermodynamic sense, as meaning a state of aggregation capable of being assumed by matter, possessing a well-defined set of physical properties such as an equation of state, a law of opacity, etc. There is no logical contradiction involved in extending the use of the word to continuous changes of phase such as are involved in the passage of a gas from its classical state to its degenerate state, for a general change of physical properties accompanies the change. The present paper is chiefly concerned with the full solution of the equations of fit of two phases against one another, for the " generalised standard model." This solution has been made possible by the aid of quadratures carried out by Mr. Fairclough at my request.

The Configurations of the Generalised Standard Model

1. The " generalised standard model " of a star is defined as a steady-state aggregate of given mass M and luminosity L, in which the source strength is uniform and equal to L/M per gram throughout the system and in which the opacity is constant in each phase, though not necessarily the

same constant in different phases. The solution of the problem of the configurations of the generalised standard model is a particular case of the fundamental problem of stellar structure, namely :

A star is observed to possess mass M *and luminosity* L*: what are the possible steady-state configurations of spherically symmetrical aggregates of matter possessing this mass* M *and endowed with sources totalling to* L *?*

To solve this problem we require a knowledge of all the different phases which matter at a high temperature is capable of assuming. These phases include the perfect-gas phase and the degenerate-gas phase, both of them in the form either predominantly non-relativistic or predominantly relativistic ; there may be and probably are other phases not yet known to us. The solution to the problem consists in enumerating in turn all the one-phase systems, the two-phase systems, the three-phase systems and so on consistent with the specification (M, L). M and L are selector variables, defining the configurations under discussion. When the enumeration has been completed as far as our present knowledge of different phases permits, comparison of the radius of the observed star (M, L) with the radii of the constructed configurations (M, L) will enable the state of the star to be diagnosed. If our knowledge of the different possible phases is complete, the constructed configurations must include amongst their number the actual configuration of the observed star (M, L).

This programme requires no knowledge of the physics of energy-generation. A knowledge of the physics of energy-generation would merely serve to reduce the labour involved, as it would direct attention to certain types of distribution of energy-sources.

If and when the identification of an observed star (M, L) with a particular one of the (M, L) configurations is accomplished, it remains to investigate why the star is in the form of one of these configurations and not the others.

It is particularly to be noted that it is not assumed that the star in nature, of mass M, emits L independently of its physical state. L is *observed*. We simply construct all the possible (M, L) configurations, *i.e.* all the configurations compatible with the possession of total mass M and total source-strength L.

It is now clear that there is no formal restriction to *observed* pairs (M, L). The programme can be carried out for *any* pair (M, L). For each mathematical pair (M, L) and for each combination of possible phases, either a configuration exists or it does not exist. Thus only certain domains of (M, L) correspond in general to steady-state configurations. If (M, L) does not lie inside the appropriate domain, a mass M endowed with sources totalling to L, restricted to the stated phases,* would either expand indefinitely, contract indefinitely or oscillate. In nature a mass M endowed with sources totalling to L and left to find its own phases could not contract indefinitely.

The programme can be carried out for any law or laws of opacity whatever.

The programme is not only a method for discovering the true nature of

* It may, of course, pass into a steady-state comprising phases other than the stated ones.

the stars. It is a necessary piece of work if we are seeking the explanation of such observed statistical characteristics as are summed up, for example, in the Russell diagram. For example, suppose the programme is completed for one-phase gaseous configurations,* for the opacity-laws found in nature and for some plausible source-distribution. Let curves of L against radius r_1 be constructed for constant M, for these configurations. To each observed star (M, L) corresponds one or more points in this diagram, and hence one or more definite radii r_1. The gaseous stars of nature, if any exist, will be represented by these points, or by a sub-set of these points, and the complete set or sub-set should immediately reproduce the Russell diagram.† For example, the (M, L) pairs lying on the main sequence should furnish the radii of the corresponding stars, *i.e.* the luminosity-colour correlation should be calculable from the luminosity-mass correlation. Again, to each observed pair (M, L) sufficiently large there correspond in nature at least two configurations, the O-type and the *g*M-type (possibly also the N-type, the S-type, etc.). If therefore the large (M, L) pairs of nature belong to gaseous stars, it must be a characteristic of constructed one-phase gaseous configurations that to each sufficiently large pair (M, L) two such configurations can be constructed of different radii. It is well known that according to the existing theory of gaseous configurations only one such configuration can be constructed, in contradiction with observation.

But quite apart from any particular applications of the method, the important point is that for any particular combination of phases the physical properties of the configurations for any pair (M, L) are completely accessible. No new knowledge of the physics of energy-generation can provide an explanation, for example, of the main features of the Russell diagram not accessible by the execution of the above programme. It can only indicate why certain M's possess certain L's. For an observed (M, L) the possible values of the radius r_1 are in principle completely calculable without any knowledge of the physics of energy-generation, for any relative source-distribution.

I have partially carried out this programme for a variety of source-distributions, and a variety of opacity-laws of the type $\kappa \propto T^m\rho^n$. The characteristic feature of the standard model, namely, that in addition to one-phase gaseous configurations there can be constructed two-phase (perfect-gas/degenerate-gas) configurations and other combinations, persists also for opacity-laws of the type $\kappa \propto T^m\rho^n$. In the (r_1, L) diagram, the curves of constant mass M, for one-phase gaseous configurations, represented for the standard model by the loci $L = L_0$, are of course deformed ‡ into more or less steep curves, undercut by the curves for collapsed or by the curves for

* As has been done by Eddington, Jeans and Russell for certain opacity-laws and source-distributions on certain boundary conditions.

† By the " Russell diagram " I mean here not Russell's original plot of L against T_e for the observed stars, but the more fundamental plot of L against r_1 for the observed stars, with the corresponding mass written against each plotted point.

‡ Mr. Greaves, of Greenwich, and Dr. Woltjer, of Leiden, have kindly helped me in elucidation of this point.

centrally-condensed two-phase configurations according to circumstances.* In A.S.S., I, I attributed the instability of gaseous configurations $L = L_0$ to the circumstances that they were *bordered* by configurations of enormously less potential energy (p. 11, bottom line; p. 26, l. 18); I was careful not to attribute it to the *non-existence* of gaseous configurations for $L \neq L_0$ which is an accidental feature of the standard model. When we pass from $\kappa =$ constant in the gaseous phase to $\kappa \propto T^m \rho^n$, from being *bordered* the gaseous configurations become *undercut* by two-phase configurations of enormously less potential energy. Hence either the instability persists, or gaseous configurations, if they can exist at all, must be metastable in the same way that an excited atom can be metastable. Should any of the stars in nature prove ultimately to be gaseous there would remain a fundamental difficulty, namely, to explain why these stars adopted a diffuse configuration when other more condensed configurations of less energy were available.

The significance of the standard model resides in the circumstance that its configurations can very largely be worked out in full, and that these may be used to yield insight into the nature of configurations for other source-distributions and other laws of opacity. The standard model is intended not to represent stars but to throw light on their structure. The possibilities which it has already yielded (in particular the existence of two-phase configurations) perhaps justify a few statements of results so far obtained, partly with the aid of calculations carried out by Mr. Fairclough.

2. As has been shown by S. Chandrasekhar † and others, the complete enumeration of the configurations of the generalised standard model includes a variety of different phase-combinations. I propose now to consider briefly two-phase systems, namely, systems consisting of a gaseous envelope of opacity $\kappa = \kappa_1$ surrounding a degenerate core of opacity $\kappa = \kappa_2$. The earlier portions of the analysis are, however, much more general and describe equally well configurations which have as their two outer zones a gaseous zone followed by a degenerate zone, whatever the state of the matter further inside.

It has been shown by S. Chandrasekhar,‡ B. Swirles§ and R. C. Majumdar || that the opacity of a degenerate gas is very small compared with what would be computed for a classical gas at the same density ρ and temperature T. This has been made clear by various investigators on simple physical grounds. In the standard model we investigate configurations in which the opacity is constant in each phase. Hence it is appropriate to select some definite constant opacity in each phase. I shall

* Since the above was written it has become clear that the locus of gaseous configurations in the (r_1, L) diagram is undercut by the locus of *collapsed* configurations *only*, both when the centrally-condensed configurations exist (M large) and when they do not (M small). When the opacity is not constant the *point* $L = L_0$, on the complete locus for a constant M, is defined as the intersection of the one-phase branch with the two-phase branch.

† *M.N.*, **91**, 456, 1931.

‡ *Proc. Roy. Soc.*, **133**, A, 241, 1931.

§ *M.N.*, **91**, 861, 1931.

|| *A.N.*, Nr. 5809, 1931.

adopt $\kappa = \kappa_1 =$ constant in the gaseous phase, $\kappa = \kappa_2 =$ constant in the degenerate phase. In an actual (M, L) configuration the opacity must be continuous across the interface, and it is in fact easy to carry out the formal analysis * with the actual laws of opacity, $\kappa_1 \propto \rho/T^{\frac{7}{2}}$, $\kappa_2 =$ const.$/T^2$. But to gain insight into the nature of the configurations it is instructive to take a discontinuous change of opacity across the phase-interface ; κ_1 and κ_2 are then given appropriate constant values. It is then consistent with physical properties of the degenerate phase to adopt $\kappa_2/\kappa_1 << 1$. In other words, when we have completed the analysis for arbitrary constant values κ_1 and κ_2 we then examine our results for the limiting case $\kappa_2 \sim 0$. The results may be expected to be indicative of conditions in the degenerate phase.

3. *The Equations of the Problem.*—The equations of the problem, due to Eddington, are

$$\frac{dp}{dr} = \frac{\kappa L(r)}{4\pi c r^2}\rho - \frac{GM(r)}{r^2}\rho, \tag{1}$$

$$\frac{d(\frac{1}{3}aT^4)}{dr} = -\frac{\kappa L(r)}{4\pi c r^2}\rho, \tag{2}$$

$$\frac{dM(r)}{dr} = 4\pi\rho r^2, \tag{3}$$

where p is the pressure, ρ the density, T the temperature, $L(r)$ the enclosed source-strength and $M(r)$ the enclosed mass, all at radius r.

Throughout the configuration the source-strength is uniform and so

$$L(r) = \frac{L}{M}M(r). \tag{4}$$

In the gas-phase we have

$$p = (R/\mu)\rho T, \qquad \kappa = \kappa_1, \tag{5}$$

and in the degenerate phase,

$$p = K\rho^{\frac{5}{3}}, \qquad \kappa = \kappa_2, \tag{6}$$

4. *The Pressure Integrals.*—By division of (1) by (2) we have

$$\frac{dp}{d(\frac{1}{3}aT^4)} = \frac{4\pi cGM}{\kappa L} - 1.$$

In any interval in which κ is constant, this integrates in the form

$$p = \tfrac{1}{3}aT^4\left(\frac{4\pi cGM}{\kappa L} - 1\right) + D, \tag{7}$$

where D is constant in the interval. D changes discontinuously from one constant value to another when κ changes. In an actual configuration in which $p = (R/\mu)\rho T$ is rapidly giving place to $p = K\rho^{\frac{5}{3}}$, D " grows " rapidly from one constant value to another.

Let D_1 be the value of D in the gas-phase, D_2 that in the degenerate phase.

* See the author's paper, *Zeits. für Astrophys.*, 4, 75, 1932.

In the gas-phase the usual external boundary condition is taken to be that $T=0$ when $\rho=0$.* This gives $D_1=0$. It follows that $D_2\neq 0$.

5. *Interfacial Conditions.*—The interface will occur at $\rho=\rho'$, $T=T'$, where

$$(R/\mu)\rho' T' = K\rho'^{\frac{5}{3}}. \tag{8}$$

Approaching the interface from the classical-gas phase we have by (7),

$$(R/\mu)\rho' T' = \tfrac{1}{3}aT'^4\left(\frac{4\pi cGM}{\kappa_1 L}-1\right). \tag{9}$$

Approaching it from the degenerate-gas phase we have similarly,

$$K\rho'^{\frac{5}{3}} = \tfrac{1}{3}aT'^4\left(\frac{4\pi cGM}{\kappa_2 L}-1\right)+D_2. \tag{10}$$

It is convenient to define β_1 and β_2 by the relations

$$\beta_1 = 1-\frac{\kappa_1 L}{4\pi cGM}, \tag{11}$$

$$\beta_2 = 1-\frac{\kappa_2 L}{4\pi cGM}. \tag{11'}$$

The solution of (8), (9), (10) is then found to be

$$\rho' = \frac{(R/\mu)^4}{\frac{1}{3}aK^3}\,\frac{1-\beta_1}{\beta_1^4}, \tag{12}$$

$$T' = \frac{(R/\mu)^{\frac{5}{3}}}{(\frac{1}{3}a)^{\frac{3}{3}}K}\left(\frac{1-\beta_1}{\beta_1}\right)^{\frac{3}{3}}, \tag{13}$$

$$D_2 = \frac{(R/\mu)^{\frac{20}{3}}}{(\frac{1}{3}a)^{\frac{5}{3}}K^4}\,\frac{(1-\beta_1)^{\frac{5}{3}}(\beta_1-\beta_2)}{\beta_1^{\frac{3}{3}}(1-\beta_2)}. \tag{14}$$

This value of D_2 ensures continuity of T at the interface provided only ρ is continuous, and it is henceforward no longer necessary to consider the temperature. We note that T' and ρ' are independent of the opacity κ_2 in the degenerate phase.

6. *Reduction to Emden's Equation.* (a) *The gas-zone.*—The reduction of (1) to Emden's equation of index 3,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^3, \tag{15}$$

for the gas-phase proceeds in the usual way. It is found that the reduction is effected by the substitutions

$$\rho = \lambda_1\theta^3, \tag{16}$$

$$r = \frac{\xi}{\lambda_1^{\frac{1}{3}}}\left[\frac{(R/\mu)^4}{\frac{1}{3}a}\,\frac{1-\beta_1}{\beta_1^4}\right]^{\frac{1}{6}}\frac{1}{(\pi G)^{\frac{1}{2}}}, \tag{17}$$

* It is this boundary condition which results in assigning arbitrary radii to one-phase gaseous configurations on the standard model. Actually the Hopf-Bronstein condition imposes a definite radius. This opens up an entirely new chapter in the theory of gaseous configurations. See *Zeits. für Astrophys., loc. cit.*

which give

$$M(r) = -\left[\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3}\frac{1-\beta_1}{\beta_1{}^4}\right]^{\frac{1}{2}}\xi^2\frac{d\theta}{d\xi}, \tag{18}$$

where λ_1 is so far undetermined. It follows that that solution of (15) is required for which, putting $M(r)=M$ in (18), at $\xi=\xi_1$,

$$\theta=0, \qquad \omega_3 \equiv -\xi_1{}^2\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} = C^{-\frac{1}{2}}, \tag{19}$$

where

$$C=\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3M^2}\frac{1-\beta_1}{\beta_1{}^4}, \tag{20}$$

and ξ_1 is arbitrary. The solution to be selected in the gas-zone thus depends solely on the physical properties in the gas-zone. If $f(\xi)$ is a solution of (15) vanishing for $\xi=\xi_0$, and satisfying $\omega_3=C^{-\frac{1}{2}}$, then

$$\theta = Af(A\xi) \tag{21}$$

is the desired solution where

$$A\xi_1=\xi_0. \tag{22}$$

The solution (21) must be inserted in (16), (17), (18).

7. *Reduction to Emden's Equation.* (b) *The Degenerate Zone.*—Without any recourse to the pressure integral (10), and accordingly *without any use of boundary conditions*,* equation (1) for the degenerate zone can be reduced as it stands to Emden's differential equation of index $\frac{3}{2}$, namely,

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left(\eta^2\frac{d\psi}{d\eta}\right) = -\psi^{\frac{3}{2}}, \tag{15'}$$

where we have put

$$\rho=\lambda_2\psi^{\frac{3}{2}}, \tag{16'}$$

$$r=\frac{\eta}{\lambda_2{}^{\frac{1}{2}}}\left[\frac{5K}{8\pi G\beta_2}\right]^{\frac{1}{2}}, \tag{17'}$$

so that

$$M(r) = -\frac{\lambda_2{}^{\frac{1}{2}}}{4(2\pi)^{\frac{1}{2}}}\left[\frac{5K}{G\beta_2}\right]^{\frac{3}{2}}\eta^2\frac{d\psi}{d\eta}, \tag{18'}$$

λ_2 being so far undetermined. Let $\phi(\eta)$ be any solution of (15'). It will be defined by the initial condition that at $\eta=\eta_0$,

$$\phi(\eta_0)=0, \qquad -\eta_0{}^5\phi'(\eta_0)=\omega_{\frac{3}{2}},$$

where $\omega_{\frac{3}{2}}$ is some constant. The general solution associated with this particular value of $\omega_{\frac{3}{2}}$ is then

$$\psi = B^4\phi(B\eta), \tag{21'}$$

and this must be inserted in (16'), (17'), (18').

8. *The Equations of Fit.*—Let the interface occur at

$$\xi=\xi', \qquad \eta=\eta',$$

* I believe that my pupil Mr. G. Lillie first drew my attention to the point mentioned in *italics*.

and put* $A\xi' = a, \qquad B\eta' = b.$

Equating the values of r' given by (17) and (17′), equating the values of $M(r')$ given by (18) and (18′), and equating to ρ' as given by (12) the values of ρ' given by (16) and (16′), we have four equations from which it will be found that we can eliminate $\lambda_1 A^3$ and $\lambda_2 B^6$. We are then left with just two independent equations, which are found to be

$$af(a) = \left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)^{\frac{1}{2}} b[\phi(b)]^{\frac{1}{4}}, \tag{23}$$

$$a^2 f'(a) = \left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)^{\frac{3}{2}} \frac{b^2\phi'(b)}{[\phi(b)]^{\frac{3}{4}}}. \tag{24}$$

By combination of these equations they may be thrown into the form

$$\frac{a[f(a)]^3}{f'(a)} = \frac{b[\phi(b)]^{\frac{3}{2}}}{\phi'(b)}, \tag{25}$$

$$\frac{af'(a)}{f(a)} = \frac{5}{8}\frac{\beta_1}{\beta_2}\frac{b\phi'(b)}{\phi(b)}. \tag{26}$$

Equation (25) puts in evidence the Emden indices 3 and $\frac{3}{2}$, and equation (26) is equivalent to the symmetrical form

$$\frac{(n_1+1)af'(a)}{\beta_1 f(a)} = \frac{(n_2+1)b\phi'(b)}{\beta_2\phi(b)},$$

with $n_1 = 3$, $n_2 = \frac{3}{2}$. Equations (25) and (26) may be said to be in canonical form. They were given by me without proof in *M.N.*, **91**, 481, 1931.

9. *Properties of Emden's Equation.*—It is convenient to have a notation in which to recapitulate certain well-known properties of Emden's equation used in the sequel. Let $\theta = f(\xi)$ be a solution of Emden's equation of index n, $\xi = \xi_0$ a zero. Then $\xi = \xi_1$ is the corresponding zero of the associated solution $\theta = A^{2/(n-1)} f(A\xi)$ if $A\xi_1 = \xi_0$. The value of

$$-\xi_1^{\frac{n+1}{n-1}}\left[\frac{d\theta}{d\xi}\right]_{\xi=\xi_1}$$

for any assigned member A of the family $\theta = A^{\frac{2}{n-1}} f(A\xi)$ is independent of the choice of A, for it is equal to

$$-\xi_0^{\frac{n+1}{n-1}} f'(\xi_0).$$

It is therefore a characteristic of the family, and its value may be taken to be the second arbitrary constant in the general solution. Calling this constant ω_n, the general solution may be written

$$\theta = A^{\frac{2}{n-1}} f(A\xi\,;\ \omega_n).$$

The value of ω_n identifies the homologous family under consideration.

* No confusion is occasioned between this a and the radiation constant, which I always write in the form $(\frac{1}{3}a)$ save in the combination L/ac. *Cf.* equation (33).

It is known * that corresponding to an assigned zero $\xi = \xi_1$ of θ, for $\xi > 0$, there exists a critical value ω_n^0 of ω_n such that :

for $\omega_n = \omega_n^0$, $\theta(\xi) \to$ a finite limit as $\xi \to 0$ and $\theta(\xi)$ has no zeros in $0 \leqslant \xi < \xi_1$;

for $\omega_n < \omega_n^0$, $\theta(\xi) \to \infty$ as $\xi \to 0$ and remains > 0 in $0 \leqslant \xi < \xi_1$;

for $\omega_n > \omega_n^0$, $\theta(\xi)$ rises to a maximum as ξ decreases from $\xi = \xi_1$, and $\theta(\xi)$ then decreases to a zero lying in $(0, \xi_1)$.

There exists likewise a monotonic increasing sequence of critical values $\omega_n^{(1)}$, $\omega_n^{(2)}$, . . . $\omega_n^{(r)}$, . . . such that for $\omega_n = \omega_n^{(r)}$, $\theta(\xi) \to$ a finite limit as $\xi \to 0$ and $\theta(\xi)$ has r zeros in $0 \leqslant \xi < \xi_1$.

When $\omega_n = \omega_n^0$, let θ_0 be the positive value to which θ tends as $\xi \to 0$, for any assigned value of A. Then

$$\theta_0 = A^{\frac{2}{n-1}} f(0 ; \ \omega_n^0),$$

whence

$$\xi_1^{\frac{2}{n-1}} \theta_0 = \xi_0^{\frac{2}{n-1}} f(0 ; \ \omega_n^0).$$

Thus $\xi_1^{\frac{2}{n-1}} \theta_0$ is the same for all members A of the homologous family $\omega_n = \omega_n^0$. It may be denoted by σ_n.

σ_n is a second invariant which is useful in the sequel. For the normalisation used in *Emden's Tables*, θ_0 is unity, and σ_n is simply $\xi_1^{\frac{2}{n-1}}$, where ξ_1 is the tabulated zero of θ.

The following values are re-calculations by Fairclough :—

$$\omega_3^0 = -\xi_0^2 f'(\xi_0 ; \ \omega_3^0) = 2{\cdot}0181 ; \quad \sigma_3 = \xi_0 f(0 ; \ \omega_3^0) = 6{\cdot}8969 ;$$
$$\omega_{\frac{1}{2}}^0 = -\eta_0^5 \phi'(\eta_0 ; \ \omega_{\frac{1}{2}}^0) = 132{\cdot}39 ; \quad \sigma_{\frac{3}{2}} = \eta_0^4 \phi(0 ; \ \omega_{\frac{1}{2}}^0) = 178{\cdot}23.$$

10. *The Physical Situation.*—We now return to the equations of fit (25) and (26). Although (25) and (26) constitute a pair of equations for the unknowns a and b, there is still one degree of freedom left in the solution, for ϕ depends on the value of $\omega_{\frac{3}{2}}$ which is so far undetermined. If the degenerate zone, $p = K\rho^{\frac{5}{3}}$, extends to the centre, $\omega_{\frac{1}{2}}$ has the value $\omega_{\frac{3}{2}}^0$ and the solution is definite. The full solution in this case will be considered shortly. It corresponds to the enumeration of all two-phase configurations. But there is no physical reason why the phase $p = K\rho^{\frac{5}{3}}$ should extend to the centre, for systems with more complicated phase-combinations may be possible. For example, the phase $p = K\rho^{\frac{5}{3}}$ may give place to the phase $p = K_2\rho^{\frac{4}{3}}$ (relativistically degenerate matter). Thus there is no necessity for $\omega_{\frac{1}{2}}$ to take the value $\omega_{\frac{3}{2}}^0$, and the complete enumeration of the configurations of the standard model must include all other possible values of $\omega_{\frac{3}{2}}$ corresponding to other phase-combinations. There is no difficulty in principle in constructing these further combinations, but in this paper I ultimately confine myself to the two-phase systems, *i.e.* to the solution of the equations of fit when $\omega_{\frac{3}{2}} = \omega_{\frac{1}{2}}^0$. The method of solution is, however, quite general and I have in fact constructed (but not completely enumerated) solutions when $\omega_{\frac{3}{2}} \neq \omega_{\frac{1}{2}}^0$.

* From the researches of R. H. Fowler and others.

11. *Physical Characteristics of the Configurations.*—Whatever value is selected for $\omega_{\frac{3}{2}}$, whenever the equations of fit possess solutions the physical state of the two outer zones becomes definite. For example, the radius of the interface and the external radius of the configuration, the mass of the core, the mean density of the core and of the whole configuration are all readily calculable. When, in addition, $\omega_{\frac{3}{2}}=\omega_{\frac{3}{2}}^{0}$, the central values of the physical variables become definite also. I propose now to tabulate the formulæ for these physical quantities. Owing to the existence of the equations of fit, transformations of these formulæ are possible and alternative formulæ can be written down. Further, the combination C may always be replaced in terms of the quantity $\xi_0^2 f'(\xi_0)$ since $C^{-\frac{1}{2}}=\omega_3=-\xi_0^2 f'(\xi_0)$. The systematisation I have adopted will be evident from the contexts.

Radius of core.

$$r'=\frac{(\frac{1}{3}a)^{\frac{1}{6}}}{(\pi G)^{\frac{1}{2}}}\frac{K}{(R/\mu)^{\frac{3}{3}}}\frac{af(a)}{\beta_1^{\frac{1}{3}}(1-\beta_1)^{\frac{1}{6}}}. \tag{27}$$

(I may mention that $(\frac{1}{3}a)^{\frac{1}{6}}K(\pi G)^{-\frac{1}{2}}(R/\mu)^{-\frac{3}{3}}$ is a fundamental length which reappears in all two-phase systems whatever the law of opacity.)

Mass of core.

$$\frac{M(r')}{M}=\frac{a^2f'(a)}{\xi_0^2f'(\xi_0)}. \tag{28}$$

Mean density of core.

$$\rho_m(\text{core})=\frac{M(r')}{\frac{4}{3}\pi r'^3}=3\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta_1}{\beta_1}\frac{-a^2f'(a)}{[af(a)]^3}. \tag{29}$$

(*Mean density of core*)/(*Interfacial density*).

$$\frac{\rho_m(\text{core})}{\rho'}=-3\frac{a^2f'(a)}{[af(a)]^3}. \tag{30}$$

Radius of configuration.

$$r_1=\frac{(\frac{1}{3}a)^{\frac{1}{6}}}{(\pi G)^{\frac{1}{2}}}\frac{K}{(R/\mu)^{\frac{3}{3}}}\frac{\xi_0 f(a)}{\beta_1^{\frac{1}{3}}(1-\beta_1)^{\frac{1}{6}}}. \tag{31}$$

Mean density of configuration.

$$\rho_m=3\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta_1}{\beta 1}\frac{-\xi_0^2f'(\xi_0)}{[\xi_0 f(a)]^3}. \tag{32}$$

Effective temperature.

$$T_e=\left(\frac{L}{ac}\right)^{\frac{1}{4}}\frac{G^{\frac{1}{2}}(R/\mu)^{\frac{1}{3}}}{(\frac{1}{3}a)^{\frac{1}{12}}K^{\frac{1}{2}}}\frac{\beta_1^{\frac{1}{6}}(1-\beta_1)^{\frac{1}{12}}}{[\xi_0 f(a)]^{\frac{1}{2}}}. \tag{33}$$

Central density. (Valid only when $\omega_{\frac{3}{2}}=\omega_{\frac{3}{2}}^{0}$.)

$$\rho_c=\frac{(R/\mu)^4}{\frac{1}{3}aK^3}\frac{1-\beta_1}{\beta_1}\frac{[\sigma_{\frac{3}{2}}]^{\frac{3}{2}}}{[\eta_0^4\phi(b)]^{\frac{3}{2}}}, \tag{34}$$

$$\frac{\rho_m}{\rho_c}=-3\frac{[\eta_0^4\phi(b)]^{\frac{3}{2}}}{[\xi_0 f(a)]^3}\frac{\xi_0^2f'(\xi_0)}{[\sigma_{\frac{3}{2}}]^{\frac{3}{2}}}. \tag{35}$$

The foregoing formulæ for physical characteristics become indeterminate as $L\to 0$, since then $a\to\xi_0$, $b\to\eta_0$, $f(a)\to 0$, $\phi(b)\to 0$, $1-\beta_1\to 0$, $C\to 0$.

The formulæ can, however, readily be put in determinate form by making use of the exact relations,

$$\frac{f(a)}{C^{\frac{1}{3}}} = \frac{5}{8}\frac{\beta_1}{\beta_2}\frac{b}{a}\left[-\frac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)} b^2\phi'(b)\right]^{\frac{1}{3}}, \tag{36}$$

$$\frac{\phi(b)}{C^{\frac{2}{3}}} = \left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)^2\left[-\frac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)} b^2\phi'(b)\right]^{\frac{1}{3}}. \tag{37}$$

The proof of these is left to the reader.

It then follows that the physical variables may be expressed in the following determinate forms:—

$$r' = \frac{5K}{2^{\frac{2}{3}}\pi^{\frac{2}{3}}GM^{\frac{1}{3}}\beta_2}\left[-\frac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)} b^5\phi'(b)\right]^{\frac{1}{3}} \tag{27'}$$

$$r_1 = \frac{5K}{2^{\frac{2}{3}}\pi^{\frac{2}{3}}GM^{\frac{1}{3}}\beta_2}\frac{\xi_0}{a}\left[-\frac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)} b^5\phi'(b)\right]^{\frac{1}{3}} \tag{31'}$$

$$\rho_m = \frac{96}{125}\frac{\pi G^3 M^2 \beta_2{}^3}{K^3}\left(\frac{a}{\xi_0}\right)^3\left[-\frac{a^2 f'(a)}{\xi_0^2 f'(\xi_0)}\frac{1}{b^5\phi'(b)}\right] \tag{32'}$$

$$T_e = \left(\frac{L}{ac}\right)^{\frac{1}{4}}\frac{2^{\frac{1}{3}}\pi^{\frac{1}{12}}G^{\frac{1}{2}}\beta_2{}^{\frac{1}{2}}M^{\frac{1}{6}}}{(5\kappa)^{\frac{1}{2}}}\left(\frac{a}{\xi_0}\right)^{\frac{1}{2}}\left[-\frac{a^2 f'(a)}{\xi_0^2 f'(\xi_0)}\frac{1}{b^5\phi'(b)}\right]^{\frac{1}{6}} \tag{33'}$$

$(\omega_{\frac{3}{2}} = \omega_{\frac{3}{2}}{}^0)$; $$\rho_c = \frac{32}{125}\frac{\pi G^3 M^2 \beta_2{}^3}{K^3}[\sigma_{\frac{3}{2}}]^{\frac{3}{2}}\left[-\frac{a^2 f'(a)}{\xi_0^2 f'(\xi_0)}\frac{(b/\eta_0)^3}{b^5\phi'(b)}\right]^2 \tag{34'}$$

$(\omega_{\frac{3}{2}} = \omega_{\frac{3}{2}}{}^0)$; $$\frac{\rho_m}{\rho_c} = -3\left(\frac{\eta_0}{b}\right)^6\frac{b^5\phi'(b)}{[\sigma_{\frac{3}{2}}]^{\frac{3}{2}}}\left(\frac{a}{\xi_0}\right)^3\frac{\xi_0^2 f'(\xi_0)}{a^2 f'(a)}. \tag{35'}$$

From these the formulæ for complete degeneracy ($L \to 0$) may be read off by inspection. It will be observed that all formulæ have been put in invariant form, *i.e.* in terms of the functions ξ/ξ_0, $\xi f(\xi)$, $-\xi^2 f'(\xi)$, η/η_0, $\eta^4\phi(\eta)$, $-\eta^5\phi'(\eta)$, so as to be independent of the "normalisation" to the zeros ξ_0, η_0.

12. *Approximate Solution of the Equation of Fit.*—Whatever the value of $\omega_{\frac{3}{2}}$, the equations of fit have an explicit approximate solution when

$$C^{\frac{2}{3}}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)$$

is small. By means of (36) and (37) it can be shown in fact that the following expansions exist :—

$$a = \xi_0\left[1 - C^{\frac{2}{3}}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)\omega_{\frac{3}{2}}{}^{\frac{1}{3}} + \ldots\right], \tag{38}$$

$$b = \eta_0\left[1 - C^{\frac{2}{3}}\left(\frac{5}{8}\frac{\beta_1}{\beta_2}\right)^2\omega_{\frac{3}{2}}{}^{\frac{1}{3}} + \ldots\right]. \tag{39}$$

These expansions afford approximate values of a and b (1) for all values of β_1/β_2 when C is small (*i.e.* for "highly collapsed" configurations); (2) for

all values of C when β_1 is small (*i.e.* for L/M near $4\pi cG/\kappa_1$).* The expressions are exceedingly useful in all investigations of the thickness of the gaseous envelope surrounding a white dwarf star, etc., but I do not pause to give all these applications here.

One application, however, is of general interest. When $\beta_1 \sim 0$, *i.e.* for L/M near its maximum value, a solution exists for which a/ξ_0 is practically equal to unity. This means that for L/M near its maximum value configurations exist for which the core contains nearly the whole of the mass. This is true whatever the value of C, *i.e.* whether the envelope is of collapsed type or of centrally-condensed type. It will appear later that such configurations are of large M, and so possess large L ; and by (33′), since L and M are large, they will therefore be of large effective temperature. It is immediately suggested that such configurations correspond to the massive highly luminous O-stars of nature. We shall see later that to any large L and M † there exists in addition a second solution, corresponding to a configuration of low effective temperature probably corresponding to the giant M-stars of nature.

The General Solution of the Equations of Fit ‡

13. *Functions associated with Emden's Equation.*—In their canonical form (25) and (26), the equations of fit may be written

$$u(a) = U(b), \tag{40}$$

$$v(a) = \frac{5}{8}\frac{\beta_1}{\beta_2} V(b), \tag{41}$$

where $u(\xi)$, $v(\xi)$, $U(\eta)$, $V(\eta)$ are defined by

$$\left.\begin{aligned} u(\xi) &= -\frac{\xi\theta^3}{\theta'}, \qquad v(\xi) = -\frac{\xi\theta'}{\theta}, \\ U(\eta) &= -\frac{\eta\psi^3}{\psi'}, \qquad V(\eta) = -\frac{\eta\psi'}{\psi}. \end{aligned}\right\} \tag{42}$$

To solve (40) and (41) it is convenient to make first some general remarks on functions associated with Emden's equation. If θ is any solution of Emden's equation of index n we can define associated functions z, w, u, v as follows :—

$$z = \xi^{\frac{2}{n-1}}\theta, \qquad w = -\xi^{\frac{n+1}{n-1}}\theta',$$

$$u = -\frac{\xi\theta^n}{\theta'} = \frac{z^n}{w}, \qquad v = -\frac{\xi\theta'}{\theta} = \frac{w}{z}.$$

We write further $x = \xi/\xi_0$;

where ξ_0 is a zero of θ. Then it is easily proved that w is the same function

* It is particularly to be noted that (38) and (39) are not the *only* solution when β_1 is small. A second solution exists when $\omega_3 < \omega_3^0$, as we shall see later.

† Within certain limits of L, for given large M.

‡ Adaptation of method first used by T. G. Cowling.

of z, and that v is the same function of u, whatever member A of a given homologous family $A^{\frac{2}{n-1}}f(A\xi)$ is selected for θ; and each is the same function of x. The same fact appears in the circumstance that w is connected with z, and v with u, by *first* order differential equations, namely:

$$\frac{dw}{dz} = -\frac{(n-3)w+(n-1)z^n}{2z-(n-1)w}, \tag{43}$$

$$\frac{u}{v}\frac{dv}{du} = -\frac{u+v-1}{u+nv-3}. \tag{44}$$

It follows that $v=v(u)$ and $w=w(z)$ are one-parameter families of curves in their respective planes for any given n, the parameter being in fact ω_n. The other constant, the constant of homology, has disappeared. It will be found that *all* physical formulæ are expressible in terms of z and w, or in terms of u and v, together with x, and thus only require the integration of a first order equation. It is readily verified that $\omega_n{}^{n-1}$ is the value of uv^n where $uv=0$.

14. The following procedure can now be used for solving (40) and (41). Plot v against u for a number of values of ω_3, and along each curve mark a "ladder" of points each labelled with the corresponding value of $x(=\xi/\xi_0)$. On the same diagram, with the same axes and on the same scale, plot V against U for a number of values of $\omega_{\frac{3}{2}}$, and along each curve mark a ladder of points each labelled with the corresponding values of $y(=\eta/\eta_0)$.

Now select, for the given L, M and κ_1, the appropriate (u, v) curve by means of the relation, $\omega_3=C^{-\frac{1}{2}}$. If the degenerate zone extends to the centre, *i.e.* if attention is confined to two-phase systems, select the (U, V) curve for which $\omega_{\frac{3}{2}}=\omega_{\frac{3}{2}}{}^0$. If the degenerate core does not extend to the centre, some other (U, V) curve may have to be selected for some other value of $\omega_{\frac{3}{2}}$ determined by the thermodynamic properties of the core. Call the (u, v) curve thus selected γ_3, the (U, V) curve $\gamma_{\frac{3}{2}}$.

Now reduce the ordinates V of the $\gamma_{\frac{3}{2}}$ curve in the ratio $(5/8)(\beta_1/\beta_2)$ and replot the curve. Call this reduced curve $\gamma_{\frac{3}{2}}'$. Then wherever $\gamma_{\frac{3}{2}}'$ intersects γ_3 we have a solution of the equations of fit. For here $u=U$, $v=(5\beta_1/8\beta_2)V$. If P is such a point of intersection, the values x' and y' of the parameters $x(=\xi/\xi_0)$ and $y(=\eta/\eta_0)$ of P may be read off from γ_3 and $\gamma_{\frac{3}{2}}'$ from the ladder of labelled points. The values of a and b then follow, for any particularisation ξ_0 and η_0 from the relations $a=\xi_0x'$, $b=\eta_0y'$. The values of all physical variables are independent of the particular normalisations ξ_0 and η_0 which identify f and ϕ for given ω_3 and $\omega_{\frac{3}{2}}$ respectively.

The complete scheme of solutions a and b can be obtained without actually drawing the intermediate curve $\gamma_{\frac{3}{2}}'$. All we have to do is to run along any curve γ_3, to any arbitrary point Q on it, of ordinate QN, and measure the corresponding ordinate Q′N of the curve $\gamma_{\frac{3}{2}}$. Then the values x' and y' corresponding to Q and Q′ afford a solution of the equations of fit for the value β_1/β_2 given by

$$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{\mathrm{QN}}{\mathrm{Q'N}}.$$

We have therefore simply to examine the families of curves γ_3 and $\gamma_{\frac{3}{2}}$ and all possible pairs of corresponding ordinates.

These families of curves can be constructed from the tables calculated by Mr. Fairclough and published in the following paper. The curves in question, as plotted by my wife, are shown in fig. 1.* Some properties of these curves are shown immediately from the differential equation (44). Each curve has a horizontal tangent where it meets the locus †

$$u+v=1,$$

and each curve has a vertical tangent where it meets the appropriate locus †

$$u+nv=3 \qquad (n=3, \quad n=\tfrac{3}{2}).$$

15.—*The curves γ_n for any index* n *are of three types :*

(1) *Collapsed Type* ($\omega_n > \omega_n{}^0$).—Each curve γ_n is asymptotic to both the u and v axes, passing from $u \sim 0$, $v \sim \infty$ to $u \sim \infty$, $v \sim 0$. Along this path YX, x decreases from unity to zero, and θ increases from its zero to its first maximum.

(2) *Emden Type* ($\omega_n = \omega_n{}^0$).—Each curve γ_n passes from $u \sim 0$, $v \sim \infty$ to the point A, ($u=3$, $v=0$). Along this path YA, x decreases from unity to zero, and θ increases from 0 to θ_0.

(3) *Centrally-condensed Type* ($\omega_n < \omega_n{}^0$).—Each curve γ_n passes from $u \sim 0$, $v \sim \infty$ to $u=0$, $v=1$ (the point B) attaining during the path a maximum value of u and a minimum value of v. Along this path YB, x decreases from unity to zero, and θ increases from 0 to ∞.

It is readily found that for all types, for $u \sim 0$, $v \sim \infty$ the asymptotic form of the curves is

$$v \sim \frac{\omega_n^{\frac{n-1}{n}}}{u^{\frac{1}{n}}}. \tag{45}$$

This follows from the asymptotic relations, valid for $t \sim 0$,

$$\theta \sim \frac{\omega_n}{\xi_0^{\frac{2}{n-1}}} t(1+t+t^2+\ldots),$$

$$u \sim \omega_n{}^{n-1} t^n,$$

$$v \sim t^{-1},$$

where

$$t=(\xi_0-\xi)/\xi_0.$$

We see that for $n=3$,

$$v \sim \omega_3{}^{\frac{2}{3}}/u^{\frac{1}{3}} \qquad (u \sim 0, \qquad v \sim \infty), \tag{46}$$

* One great merit of the (u, v) curves is that the scale of ξ is enormously pulled out along the centrally-condensed curves just where ξ is small, *i.e.* in the interesting region. As it were, the transformation from the (ξ, θ) plane to the (u, v) plane applies a microscope where ξ is small.

† These loci are shown as dotted lines in fig. 1, for $n=3$ and $n=\frac{1}{2}$.

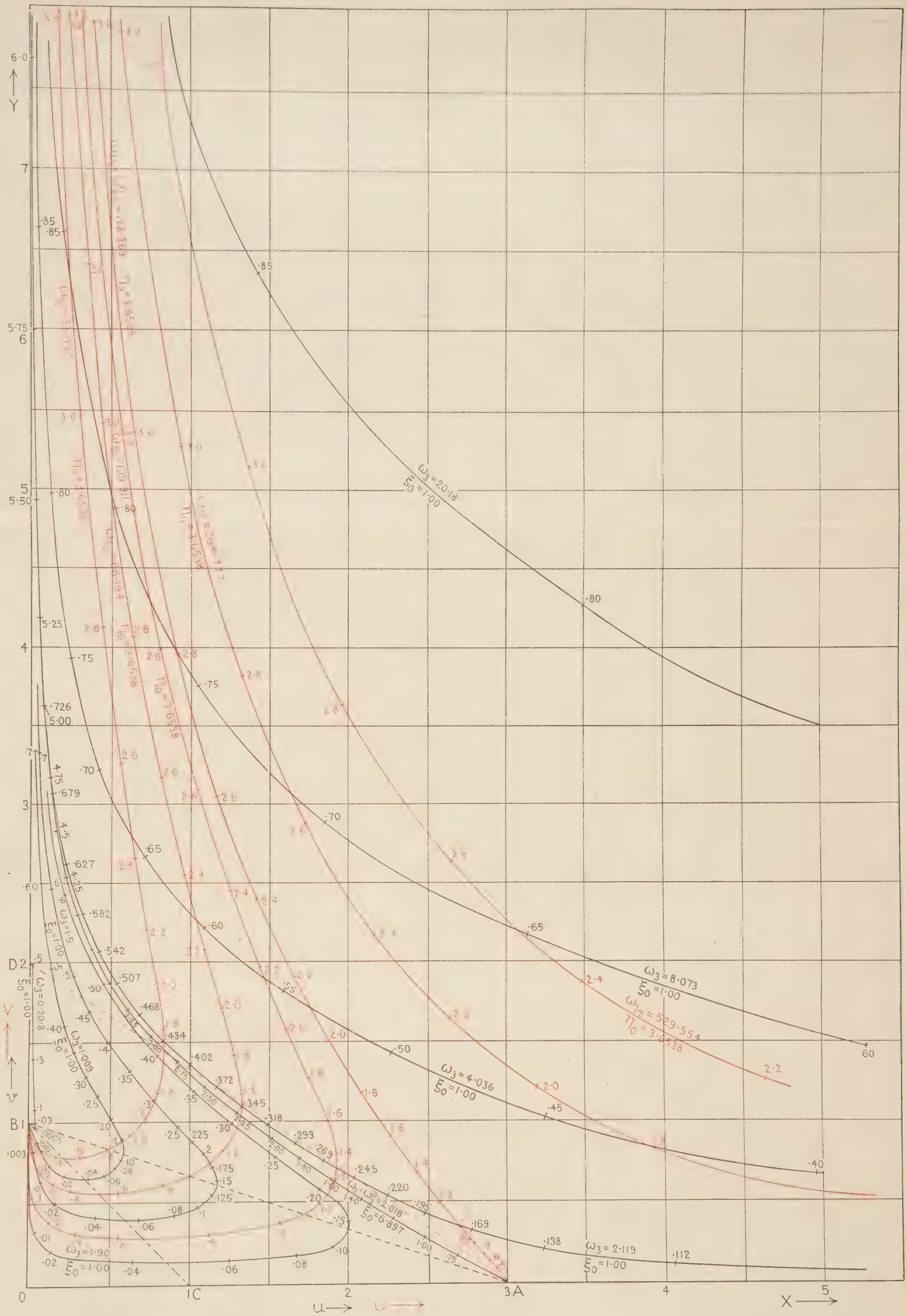

FIG. 1.—*The black curves show* $v(\xi)$ *plotted against* $u(\xi)$ *for different values of* ω_3. *The small figures in black along each curve give the values of* ξ, *and the value of* ξ_0 *is also given. The red curves show* $V(\eta)$ *plotted against* $U(\eta)$ *for different values of* $\omega_{\frac{3}{2}}$. *The small figures in red along each curve give the values of* η, *and the value of* η_0 *is also given. The locus of horizontal tangents and the loci of vertical tangents are shown as dotted lines.*

and for $n=\frac{3}{2}$,

$$V \sim \omega_{\frac{3}{2}}^{\frac{1}{3}}/U^{\frac{2}{3}} \qquad (U \sim 0, \qquad V \sim \infty), \tag{47}$$

whence it follows that when $u=U$, for all sufficiently small u, near $v \sim \infty$, $V \sim \infty$ we have $V > v$. Thus every curve $\gamma_{\frac{3}{2}}$ ultimately ascends, near v, $V \sim \infty$, above each curve γ_3.

Near the point A the curves γ_n are readily found to possess the approximate formula *

$$v^{\frac{3}{2}}[5(3-u)-3nv] = \text{constant} = \tau, \tag{48}$$

and so resemble portions of generalised hyperbolas, asymptotic to $v=0$ and $5(3-u)-3nv=0$, the value of the constant τ for given n being a function of ω_n. The constant τ is positive for $\omega_n > \omega_n{}^0$, negative for $\omega_n < \omega_n{}^0$. When $\omega_n = \omega_n{}^0$, *i.e.* for the Emden curve, the value of the constant τ is zero, and the locus is represented near A by its tangent at A,

$$5(3-u)-3nv=0,$$

together with the straight line $v=0$.

16. *Enumeration of Solutions.*†—Let us now fix attention on two-phase configurations, for which the degenerate zone extends to the centre. Then $\omega_{\frac{3}{2}} = \omega_{\frac{3}{2}}{}^0 = 132{\cdot}39$, and the corresponding curve $\gamma_{\frac{3}{2}}{}^0$ is of the form YA.

Any Emden curve $\gamma_n{}^0$ starts at A with a negative slope $-dv/du = 5/3n$; the parametric representation near A, in the usual Emden normalisation, being

$$\left.\begin{aligned} \theta &\sim 1 - \frac{\xi^2}{6} + \frac{n}{120}\xi^4 \\ u &\sim 3 - \frac{n\xi^2}{5} \qquad (\xi \sim 0) \\ v &\sim \xi^2/3 \end{aligned}\right\}. \tag{49}$$

Thus $\gamma_{\frac{3}{2}}{}^0$ starts at A with a negative slope 10/9, whilst $\gamma_3{}^0$ starts with a negative slope 5/9. We now consider in turn the solutions for the different types of configurations.

(a) *Collapsed Configurations* ($C^{-\frac{1}{3}} > \omega_3{}^0$)

We denote by $(\gamma_{\frac{3}{2}}{}^0)'$ the curve obtained by reducing the ordinates of $\gamma_{\frac{3}{2}}{}^0$ in the ratio $(5/8)(\beta_1/\beta_2)$. Now the curve $(\gamma_{\frac{3}{2}}{}^0)'$ starts at A below any curve γ_3 of the type $\omega_3 > \omega_3{}^0$ and certainly passes above it. It is clear by inspection of the diagram that each γ_3 is intersected by $(\gamma_{\frac{3}{2}}{}^0)'$ in one and only one point whatever the reduction factor $(5/8)(\beta_1/\beta_2)$. It follows that, *when the gaseous envelope is of collapsed type, the equations of fit possess one and only one solution a, b for every value of* β_1/β_2. Hence to every L, M, κ_1 for which $C^{-\frac{1}{3}} > \omega_3{}^0$ there exists one and only one configuration.

* This formula was obtained by Prof. R. H. Fowler and myself during a discussion. I am, as ever, indebted to Prof. Fowler for his lively interest in these problems.

† The results of § 16 were announced in a preliminary fashion in *Observatory*, 54, 247, 1931.

(b) *Quasi-diffuse Configurations* ($C^{-\frac{1}{2}} = \omega_3{}^0$)

Consider now the curve $\gamma_3{}^0$. The negative slope of $(\gamma_{\frac{3}{2}}{}^0)'$ at A is

$$\frac{10}{9} \times \frac{5}{8} \frac{\beta_1}{\beta_2} = \frac{25}{36} \frac{\beta_1}{\beta_2},$$

and it steadily increases as u decreases. The negative slope of $\gamma_3{}^0$ is 5/9. The curve $(\gamma_{\frac{3}{2}}{}^0)'$ accordingly starts off below $\gamma_3{}^0$ only if $(25/36)(\beta_1/\beta_2) < 5/9$, *i.e.* if $\beta_1/\beta_2 < 4/5$. Further, $(\gamma_{\frac{3}{2}}{}^0)'$ ultimately ascends above the curve $\gamma_3{}^0$. It follows by inspection of the diagram that, if

$$\beta_1/\beta_2 > 4/5,$$

$(\gamma_{\frac{3}{2}}{}^0)'$ and $\gamma_3{}^0$ have no intersections other than A ; if

$$\beta_1/\beta_2 = 4/5$$

they touch at A, and have no other point of intersection ; if

$$\beta_1/\beta_2 < 4/5$$

they have one and only one point of intersection other than A. Hence *when the gaseous envelope itself is of diffuse type* ($C^{-\frac{1}{2}} = \omega_3{}^0$), *in addition to the solution* $a = 0$, $b = 0$, *there is no solution when* $\beta_1/\beta_2 \geqslant 4/5$, *one solution when* $\beta_1/\beta_2 < 4/5$.

When, for $C^{-\frac{1}{2}} = \omega_3{}^0$, a solution exists other than $a = 0$, $b = 0$, the corresponding configuration may be described as *quasi-diffuse*. The gaseous envelope is of diffuse type, and would if continued to the centre reach it without singularity ; but the gas-laws break down first and yield a degenerate core extending to the centre.

It follows that when $C^{-\frac{1}{2}} = \omega_3{}^0$, two entirely different types of configurations are possible. There are first the wholly diffuse configurations, consisting of Emden-Eddington polytropes, of arbitrary radius. The limiting member of these corresponds to $a = 0$, $b = 0$ and has incipient central degeneracy. There is, secondly, a set of configurations (to me entirely unexpected) consisting of a gaseous envelope of diffuse type surrounding a degenerate core ; these exist whenever $\beta_1/\beta_2 < 4/5$. It is particularly to be noted that imposition of the condition $C^{-\frac{1}{2}} = \omega_3{}^0$ does not ensure a complete gaseous configuration. This condition merely ensures an envelope of diffuse type. Quasi-diffuse configurations ($\beta_1/\beta_2 < 4/5$) possess the luminosity $L = L_0$ of my previous paper, but are composite.

(c) *Centrally-condensed Configurations* ($C^{-\frac{1}{2}} < \omega_3{}^0$)

When the reduction factor $(5/8)(\beta_1/\beta_2)$ is sufficiently small the curve $(\gamma_{\frac{3}{2}}{}^0)'$ crosses the lobe of the curve γ_3, passes within it and ultimately ascends very steeply, thus emerging from the lobe and intersecting it in a second point. For this to occur it is necessary, for any given centrally-condensed γ_3, that (β_1/β_2) should be less than some number i, which is clearly such that

$$\tfrac{10}{9} \times \tfrac{5}{8} i < \tfrac{5}{9}$$

or

$$i < 4/5.$$

For given γ_3, i depends on the value of ω_3, and is smaller the smaller is ω_3. We have then the following result: *for each ω_3 less than ω_3^0 there exists a number i ($< \frac{4}{5}$), a function of ω_3, such that*

when $\beta_1/\beta_2 > i$, the equations of fit have no solution,
when $\beta_1/\beta_2 = i$, the equations of fit have two coincident solutions,
when $\beta_1/\beta_2 < i$, the equations of fit have two and only two distinct solutions.

Thus when the envelope is of centrally-condensed type, either two configurations are possible, or two coincident configurations, or no configuration.

It is easily seen that for β_1/β_2 very small, the two solutions are respectively $a \sim 0$ and $a \sim \xi_0$, *i.e.* the first configuration has a very small core, the second is nearly all core. The second one has already been encountered in discussing an approximate form of the solution of the equations of fit; it was there held to represent the structure of the O-stars of nature. The first configuration similarly is possibly the counterpart of the giant red stars of nature, for these are nature's second type of configuration of large M and L. The existence of two distinct solutions of the equations of fit, when the envelope is of centrally-condensed type, and β_1/β_2 is not too large,* is a fundamental characteristic of the generalised standard model. I suggest that it is the counterpart, for this model, of the existence in nature of two distinct types of configuration for sufficiently large M and L, namely, the O-type stars and the giant red stars.

17. *The Solution-array.*—The array of solutions a for all C and all values of β_1/β_2 is shown graphically in fig. 2. Here the relative core-radius a/ξ_0 is shown plotted against β_1/β_2, for curves of constant ω_3, *i.e.* for constant C. The diagram is lettered to correspond to fig. 1, each curve γ_3 in fig. 1 having a counterpart in fig. 2. The curves have been constructed from tables of solutions of the equations of fit prepared in the manner described.

It will be seen that quasi-diffuse two-phase configurations are represented by the curve AY, where A is the point

$$\beta_1/\beta_2 = 4/5, \qquad a/\xi_0 = 0,$$

Y the point

$$\beta_1/\beta_2 = 0, \qquad a/\xi_0 = 1.$$

To each value of β_1/β_2 less than 4/5 corresponds one point in the curve AY. Collapsed configurations are represented by curves passing through Y and becoming ultimately parallel to BAX (for convenience the diagram is artificially cut off by the ordinate $\beta_1/\beta_2 = 1$). Centrally-condensed configurations are represented by curves passing from Y to B inside YAB. We see at once that for $\beta_1/\beta_2 > 4/5$ the only configurations possible are collapsed ones. In particular, for $\beta_1 = \beta_2$ the only configurations are collapsed ones. We thus derive the results of Cowling,† Strömgren ‡ and Russell §

* We shall see shortly that the two conditions $C^{-\frac{1}{2}} < \omega_3^0$ and $\beta_1/\beta_2 < i(\omega_3)$ imply that M must be sufficiently large ($M > \mathfrak{M}$) and that L must only slightly exceed L_0.

† *M.N.*, **91**, 472, 1931.

‡ *Ibid.*, **91**, 466, 1931.

§ *Ibid.*, **91**, 739, 1931.

as a very particular case. For $\beta_1/\beta_2 < 4/5$, collapsed, centrally-condensed and quasi-diffuse configurations can all be constructed.

18. *Curves of Constant Mass.*—The values of ω_3 and of β_1/β_2 may be

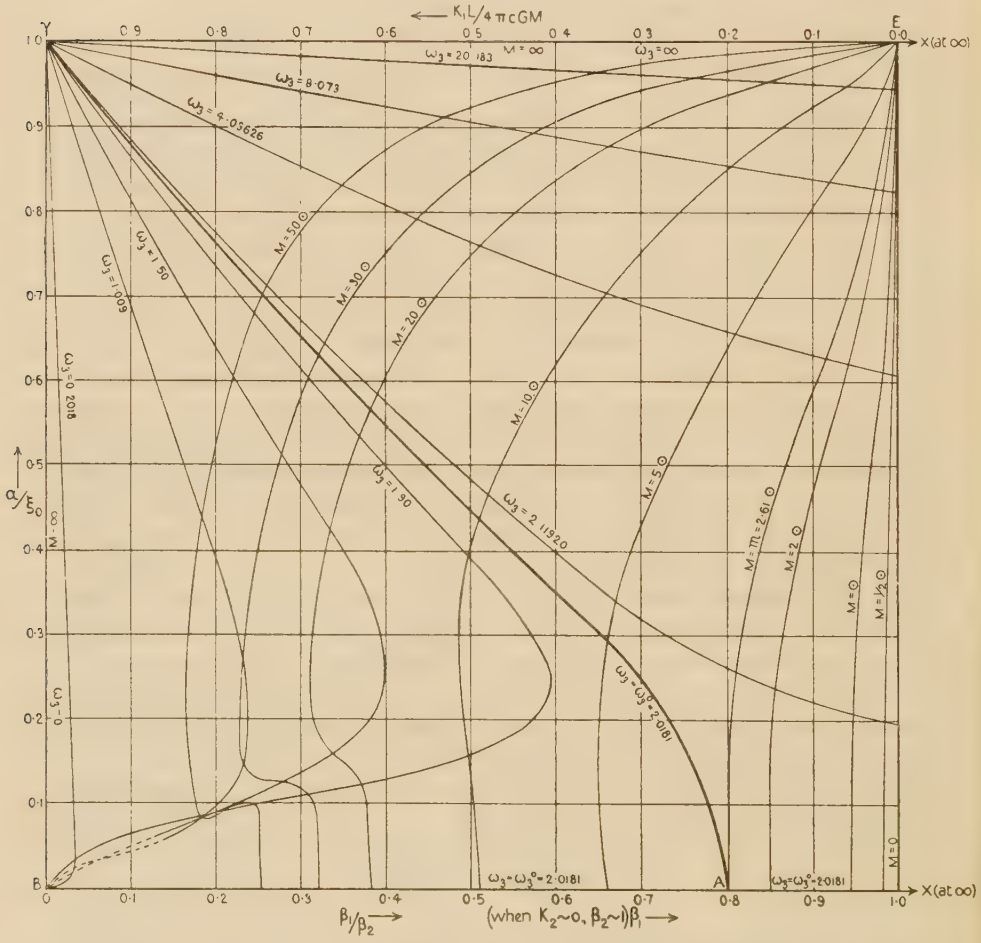

FIG. 2.—*Curves of relative core-radius* a/ξ_0 *plotted against* β_1/β_2 *for constant* ω_3, *from* $\omega_3 = 0$ *to* $\omega_3 = \infty$. *Also (in the case* $\beta_2 \sim 1$, $\kappa_2 \sim 0$*) curves of constant mass* M *from* M = 0 *to* M = ∞ *; the scale of* β_1 *may be regarded in this case as a scale of* L/M *running from right to left.*

taken as a co-ordinate system capable of locating the position of any point in fig. 2. Since

$$\omega_3 = C^{-\frac{1}{2}} = \left[\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3M^2}\,\frac{1-\beta_1}{\beta_1^4}\right]^{-\frac{1}{2}},$$

$$\beta_1/\beta_2 = \frac{\beta_1}{1-(\kappa_2/\kappa_1)(1-\beta_1)},$$

to every point $(\omega_3,\ \beta_1/\beta_2)$ in fig. 2 there corresponds one or more values of the pair $(M,\ \beta_1)$ obtained by solving the two preceding equations for M and

β_1 for any given value of (κ_2/κ_1). By joining up points of constant mass M, we can inscribe on the diagram curves of constant mass M, for any given (κ_2/κ_1), and so examine the different configurations of a given mass M for various luminosities, when constrained to two-phase systems of given κ_2/κ_1. In other words, fig. 2 contains implicitly the solution of the fundamental problem of stellar structure for the generalised standard model; for it provides the behaviour of the relative core-radius a/ξ_0 for prescribed M for all *possible* values of L. The external radius follows from (31).

19. *Mass-curves for* $\kappa_2 \sim 0$.*—In order to gain insight into the results for any value of κ_2/κ_1, it seems best to examine in detail the limiting case of zero opacity in the core, namely, the case $\kappa_2 \sim 0$, $\beta_2 \sim 1$. That κ_2 is absolutely very small in a degenerate region as compared with κ_1 in a gaseous region we have seen to be given by the physical theory of opacity. The limiting case $\kappa_2/\kappa_1 \sim 0$ is therefore of great interest.

We have now that β_1/β_2 reduces to β_1; and since C is a function of M and β_1, to each pair (ω_3, β_1) corresponds a definite value of M. The form of the loci of constant mass M in the $(\beta_1, a/\xi_0)$ diagram is now readily obtained.

In fig. 2, each collapsed ω_3-curve $(\omega_3 > \omega_3^0)$ meets each ordinate $\beta_1 =$ constant in one and only one point. To this point on the assigned ω_3-curve corresponds a definite value of M, and since all other points on the same ω_3-curve correspond to different values of β_1, they correspond to different values of M. Hence any curve of constant M meets the ω_3-curve in at most one point. But each centrally-condensed ω_3-curve $(\omega_3 < \omega_3^0)$ meets each ordinate $\beta_1 =$ constant (provided $\beta_1 < i(\omega_3)$) in two points; and at these two points, since C and β_1 take the same values, M takes the same value. Hence if any M-curve meets a centrally-condensed ω_3-curve in a point S, it also meets it in a second point T on the same ordinate. Again, if a mass-curve meets the quasi-diffuse locus $\omega_3 = \omega_3^0$ in a point S′, it meets the axis of β_1 in a point T′ vertically below S′.

Some typical mass-curves, constructed by actual calculation, are shown in the diagram, fig. 2. The same curves are replotted in fig. 3 with circles indicating roughly the relative radii of the corresponding configurations. †

20. *Properties of the Mass-curves for* $\kappa_2 \sim 0$.—The horizontal scale of β_1 may now be regarded as a scale of L/M in the negative direction. The point E corresponds to $L/M = 0$; and the configurations for the largest possible values of L/M will occur in the region near the left-hand edge of the diagram.

It is readily found that each mass-curve starts from the point E $(a/\xi_0 = 1$, $\beta_1 = 1$, $L/M = 0)$ and passes to the β_1-axis. For small values of M it remains entirely inside the collapsed region $(\omega_3 > \omega_3^0)$, crossing each ω_3-curve once

* When $\kappa_2 \sim 0$, the temperature of the core is practically T', as given by (13) above. The important physical consequences of the isothermal character of the core have been developed by the author in *Nature*, 1931 December 12.

† The radius-scale of the circles is not the same for different mass-curves. For example, each mass-curve converges on the point E with a different radius, given by $r_1 \propto M^{-\frac{1}{3}}$ (formula for a completely degenerate configuration).

and once only. For large values of M it crosses all the collapsed ω_3-curves once and once only, then crosses the quasi-diffuse curve $\omega_3 = \omega_3^0$, enters the centrally-condensed region and crosses each centrally-condensed ω_3-curve

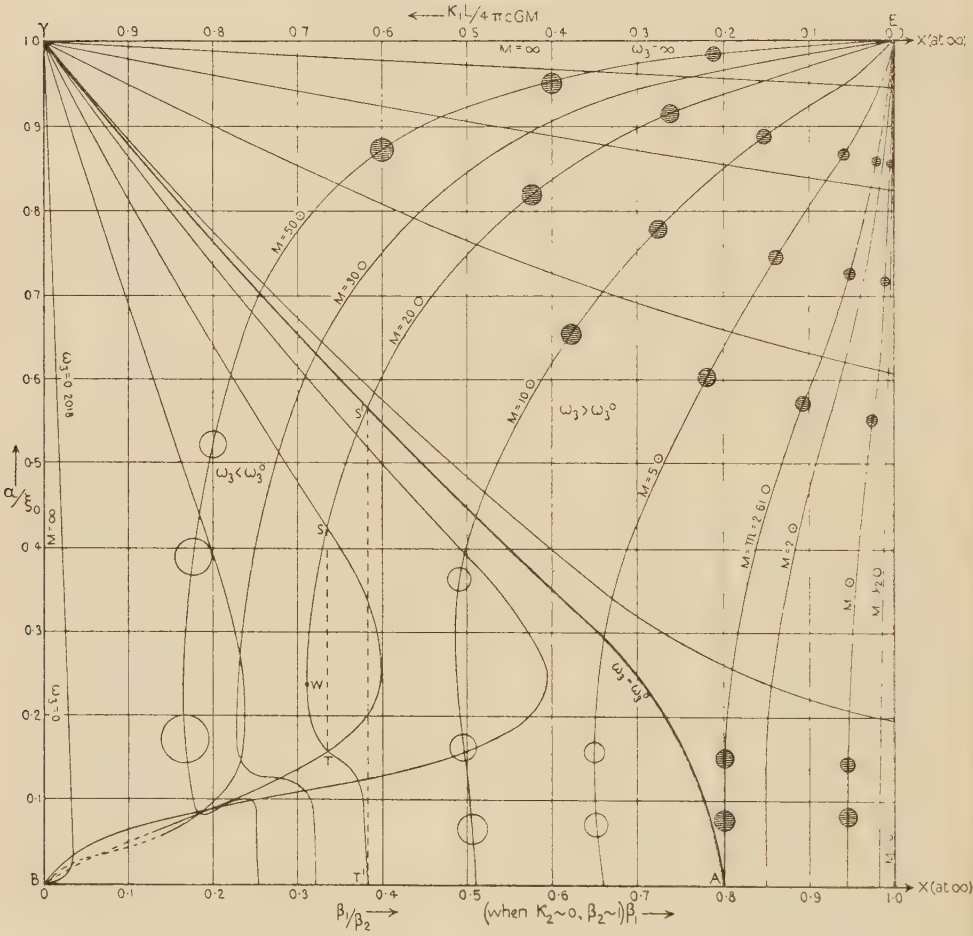

FIG. 3.—*Curves of relative core-radius* a/ξ_0 *plotted against* β_1/β_2 *for constant* ω_3. *Also curves of* a/ξ_0 *against* β_1 *for constant mass* M *when* $\kappa_2 \sim 0$, $\beta_2 \sim 1$. *The locus* $\omega_3 = \omega_3^0$ *divides "centrally-condensed" configurations from "collapsed" configurations, and itself gives the quasi-diffuse configurations. The circles indicate* qualitatively *the external radii of the configurations; they are not drawn on a uniform radius-scale. Open circles denote "centrally-condensed" configurations, shaded circles "collapsed" configurations. Points such as W, where* $(\partial a/\partial \beta_1)_M = \infty$, *represent configurations probably possessing Cepheid characteristics.*

twice and twice only. For very large values of M the mass-curve consists of a nearly horizontal portion and a nearly vertical portion mostly convex to the axis of a/ξ_0.

The mass-curve which separates the two types is readily found. Since it passes through the point A ($\beta_1 = \frac{4}{5}$, $\omega_3 = \omega_3^0$), it is the curve $M = \mathfrak{M}$, where

$$\omega_3^0 = 2{\cdot}0181\ \ldots = \left[\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3\mathfrak{M}^2}\,\frac{1-\beta_1}{\beta_1^4}\right]_{\beta_1=\frac{4}{5}}$$

which yields

$$\frac{\mathfrak{M}}{\odot} = 2{\cdot}62\left(\frac{2{\cdot}2\mu_H}{\mu}\right)^2. \tag{50}$$

(μ_H/μ) is the ratio of mass of the proton to the mean mass of the particles.

It follows that :

(1) For $M \leqslant \mathfrak{M}$, all configurations which exist are of collapsed type, and β_1 necessarily lies between $\beta_1{}^0$ and unity, where $\frac{1}{3} < \beta_1{}^0 < 1$, $\beta_1{}^0$ being a function of M given by

$$\omega_3{}^0 = 2{\cdot}0181 \ \ldots = \left[\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3M^2}\,\frac{1-\beta_1{}^0}{(\beta_1{}^0)^4}\right]^{-\frac{1}{2}}.$$

(2) For $M > \mathfrak{M}$, the configurations which exist are either collapsed $(\beta_1 > \beta_1{}^0)$ or centrally-condensed $(i < \beta_1 < \beta_1{}^0)$, where i is the quantity introduced previously, calculated for that value of ω_3 for which

$$\omega_3 = \left[\frac{16(R/\mu)^4}{\pi(\frac{1}{3}a)G^3M^2}\,\frac{1-i}{i^4}\right]^{-\frac{1}{2}}.$$

(The value of ω_3 which is the root of this equation determines the ω_3-curve which touches the mass-curve in question, on inserting i as a function of ω_3.) We see that i can now be regarded as a function of M only. For $\beta_1{}^0 < \beta_1 < 1$ there is one possible configuration ; for $i < \beta_1 < \beta_1{}^0$ there are two possible configurations.

The value $\beta_1 = i(M)$ determines the maximum value of L/M for which configurations exist. It gives

$$\left(\frac{L}{M}\right)_{\max} = \frac{4\pi cG(1-i(M))}{\kappa_1}. \tag{51}$$

Call this quantity $\epsilon_{\max}(M)$, ϵ itself denoting L/M. It follows that for

$$\epsilon_{\max}(M) > \epsilon > \frac{4\pi cG(1-\beta_1{}^0)}{\kappa_1}$$

two configurations exist, of different relative core-radii, and so of different external radius. For

$$\frac{4\pi cG(1-\beta_1{}^0)}{\kappa_1} > \epsilon > 0$$

one configuration exists.

Thus, as ϵ decreases below $\epsilon_{\max}(M)$, for given M, two series of configurations exist. In the one, of the larger relative core-radii, the core-radius steadily increases until the core absorbs the whole configuration, when we have a completely degenerate mass. In the other, of the smaller core-radii, the core-radius steadily decreases to zero. The two series are represented by WSS′E and WTT′. They pass continuously into each other at W, and along WSS′E the centrally-condensed configurations are continuous at S′ with the collapsed configurations. At T′, the core-radius being zero, the configuration takes the form of an Emden-Eddington polytrope.*

* I am not sure whether or no the radius of the configurations is continuous with the series of Emden-Eddington polytropes near T′.

21. *Nova Phenomenon.*—What happens along WTT′ as the representative point reaches T′, ϵ decreasing? This is a highly interesting and important point. The only two-phase configurations possible are represented by TE, and accordingly, if for any reason configurations are restricted to at most two-phase configurations, the only possibility is that the representative point P jumps discontinuously from T′ to T, and the core-radius increases discontinuously. Physically the course of events is obvious. With waning energy-generation light-pressure becomes inadequate to support the gaseous envelope; some of the envelope condenses in a core, but the condensation of this portion causes a further collapse of the envelope, and so on. Equilibrium is re-established when the relative core-radius has grown from zero to the value T′T. A large amount of gravitational energy will be released in this process and the system will exhibit the phenomena of a nova, namely, temporary brightening and cataclysmic decrease of external radius.*

22. We have thus the following two fundamental properties of the two-phase configurations of the generalised standard model :—

(1) For $M > \mathfrak{M}$, there exist values $\epsilon_{\max}(M)$ and $\beta_1{}^0(M)$, $(< \frac{4}{5})$, such that for

$$\epsilon > \epsilon_{\max}(M)$$

no configurations exist; for

$$\epsilon_{\max}(M) > \epsilon > \frac{4\pi cGM(1-\beta_1{}^0)}{\kappa_1}$$

two configurations exist; for

$$\frac{4\pi cG(1-\beta_1{}^0)}{\kappa_1} > \epsilon > 0$$

one configuration exists. The configurations for given M may be arranged in two linear series. Along the one, as ϵ decreases from $\epsilon_{\max}(M)$ to zero the relative core-radius a/ξ_0 steadily increases from a certain value a'/ξ_0 to unity. Along the other, the relative core-radius a/ξ_0 steadily decreases from a'/ξ_0 to zero, and then (at $\epsilon = 4\pi cG(1-\beta_1{}^0)/\kappa_1$) increases cataclysmically, the configuration rejoining the other series (nova phenomenon).

(The first series resembles the O-stars and nuclei of planetary nebulæ of nature, for it consists of massive configurations of relatively large core and so of relatively small external radius. The second series resembles the giant red stars of nature, for it consists of massive configurations of relatively small core and so of large external radius.)

(2) For $M \leqslant \mathfrak{M}$ there exists a number $\beta_1{}^0(M)$, $(> \frac{4}{5})$, such that for

$$\epsilon > \frac{4\pi cG(1-\beta_1{}^0)}{\kappa_1}$$

no configurations exist; for

$$\frac{4\pi cG(1-\beta_1{}^0)}{\kappa_1} > \epsilon > 0$$

* It is not, of course, supposed that in nature a representative point would remain on the same mass-curve. The representative point would travel from curve to curve as M decreased. The " nova-phenomenon " would occur whenever it reached a point such as T′.

one configuration exists. The configurations for given M may be arranged in a linear series. Along the linear series, as ϵ decreases to zero the relative core-radius increases from zero to unity.

(The small-cored configurations of these series will resemble the dwarf main sequence stars of nature ; for they will have small mass, small luminosity and not too small radii. Those which possess larger cores will resemble the white dwarfs of nature, for they will have small mass, lower luminosities for given mass and small radii.)

23. *Mass-luminosity Correlation.*—The two-phase configurations we have constructed possess many other striking properties. We saw that for $M > \mathfrak{M}$ the centrally-condensed portions of the mass-curves depart only slightly from the vertical ; they lie between $\beta_1 = i$ and $\beta_1 = \beta_1^0$, and i and β_1^0 are very close together. It follows that the permissible limits of ϵ are close together. In other words, L must lie between narrow limits, depending on M, for a centrally-condensed configuration to be possible. If L lies above these limits, no configuration exists, and the system would either burst or cease to be a two-phase system.* If L lies below them, the system is collapsed with a large relative core-radius. We thus see that for $M > \mathfrak{M}$ the configurations which are centrally-condensed will exhibit an approximate, but not exact, mass-luminosity correlation.

When $M \leqslant \mathfrak{M}$ the lower parts of the mass curves are also nearly vertical, and so the smaller-cored collapsed configurations will also exhibit an approximate mass-luminosity correlation. Since the luminosity is in each case given approximately by

$$L \doteq \frac{4\pi cGM(1-\beta_1^0)}{\kappa_1}, \tag{52}$$

it follows that configurations in question, both for $M > \mathfrak{M}$ and for $M \leqslant \mathfrak{M}$, will all obey the same mass-luminosity correlation. Thus the configurations we have held to resemble the B, O-stars, the giant red stars and the main sequence of stars of nature all obey the same approximate mass-luminosity correlation. The configurations we have held to resemble the nuclei of planetary nebulæ and the white dwarfs will depart from this correlation in the sense of possessing arbitrarily smaller luminosities for given mass. The configurations resembling main sequence stars may possess luminosities somewhat less than that given by (52) but not greater ; those resembling B, O-stars and giant red stars will possess luminosities somewhat greater than that given by (52).

The wheel has now come full circle. We saw at the beginning of this series of investigations, as was first pointed out by Jeans, that L and M are independent variables and that L cannot exceed a certain limit depending on M.† We now find that for configurations possessing certain well-marked physical characteristics, L must lie between narrow limits depending on M;

* Presumably a point such as W is a point of bifurcation, where a series of *three-phase* systems commences.

† That is, for two-phase configurations. I have not yet systematically investigated three-phase configurations.

if L exceeds these limits the system will burst ; * if L falls below these limits the configuration will necessarily possess a large relative core-radius. I suggest that the study of two-phase configurations has revealed in principle the origin of nature's mass-luminosity correlation. The current theory of *gaseous* configurations of opacity $\kappa \propto \rho/T^{7/2}$ permits, on the contrary, a given M to have any L, arbitrarily large or small, by adjustment of radius, in accordance with the formula $r_1 \propto L^{-2}f(M)$. The explanation of mass-luminosity correlation we have encountered in this paper is novel and unexpected. It may of course require modification when three-phase configurations come to be considered, as some of our constructed (M, L) two-phase configurations will be absorbed into the enumeration of three-phase ones.

24. *Cepheid Phenomenon.*—Another striking property of the two-phase configurations arises when we consider in fig. 3 a point such as W, where $\partial a/\partial\beta_1 = \infty$. The configuration represented by W has the maximum possible luminosity for its mass. Its core-radius is indefinite, and capable of infinitesimal alteration for the same L. Such a configuration would fluctuate in core-radius, and so in external radius. Such fluctuations would cause physical changes in the internal state of the configuration—changes outside the scope of this paper—and these would presumably produce changes in energy-generation, which would induce further oscillations superposed on the original fluctuations. It is easy to see in a general way that the result would be to set the configuration pulsating.† We have in fact a configuration favourable for exhibiting the phenomena of Cepheid variation. This suggested explanation of the origin of Cepheid variation is confirmed by the circumstance that the possibility of these fluctuations arises only when $M > \mathfrak{M}$; moreover, for the greatest possible luminosity for this M. Further, these configurations have an intermediate relative core-radius—that is to say, for the given M and maximum L they possess a core-radius intermediate between the core-radii of the two possible configurations for any slightly smaller L. Hence they will have intermediate external radii. This strikingly suggests the observed occurrence of Cepheid variation round about types A to G, intermediate between type gM and type B, O. We may say that two-phase configurations on the generalised standard model exhibit the phenomena of pseudo-Cepheid variation for $M > \mathfrak{M}$ when at the maximum luminosity for this M.‡

* Or generate a third phase.

† The nature of the pulsation would require exhaustive consideration, as restoring forces would only be called into play by the second-order effects, if the mass M and generation of energy L remained constant. First-order displacement would leave the configuration still one of equilibrium. Thus quite large amplitudes of oscillation will be compatible with small damping.

‡ A configuration represented by a point such as W has the maximum possible control temperature T' or T_c and the maximum possible interfacial temperature ρ' for the given mass M ; for here β_1 has its least possible value. Thus if temperature encourages energy-generation, the physical conditions corresponding to W are likely to be compatible with its other property of having the highest value of L/M for the given mass M. In other words, Cepheids have the highest possible central temperature for the given mass.

25. *Massive Red Configurations.*—Lastly, the mass-curves for large M in the region of small core-radii exhibit curious kinks, possessing two points where $\partial\alpha/\partial\beta_1 = 0$, dividing three distinct branches. This is also the region of large external radii and large luminosity, and there is evidently scope here for the existence of configurations possessing various kinds of physical properties. It is precisely in this same region of large M, large L and large external radius that nature exhibits a variety of configurations, namely, M-type giants, N-type giants, S-type giants and M-type variables. A great deal of calculating and thinking requires to be done to elucidate the curious state of affairs in the bottom left-hand corner of fig. 3. Here we are content to leave it as a hopeful field for future research.

26. *Contrast with the Current Theory of Gaseous Configurations.*—The *existing* theory of *gaseous* configurations * offers a set of configurations in complete contrast to those just discussed. As is well known, it predicts substantially, for $\kappa_1 \propto \rho/T^{\frac{7}{2}}$, the proportionality (Eddington) †

$$r_1 \propto L^{-2} f(M), \tag{53}$$

where $f(M)$ is a single-valued function of M. It thus predicts (*a*) a unique configuration for every pair L, M; (*b*) a value r_1 decreasing as L increases. Result (*a*) is in contradiction to the observed existence of configurations of the same L and M but of different radius. The examples of Capella and Algol have been mentioned by Eddington,‡ but more conspicuous examples are offered by the existence of B, O-type stars and of giant M-type stars of the same large L and M. It seems difficult to reconcile the existing theory of one-phase gaseous systems with the existence of these pairs.§ Two-phase configurations, however, possess this property for $M > \mathfrak{M}$ and L sufficiently large. Again, result (*b*) is very hard to reconcile with observation, as it should permit the existence of very large stars of small luminosity; giantism should be, according to it, a property of low-luminosity stars of large mass. The proportionality (53) seems to be irreconcilable with the observed characteristic of the stars as summed up in the Russell diagram. Again, since (53) predicts increasing r_1 for decreasing L, for constant M, according to (53), the most luminous configuration for given M should tend first to deviate from the state of a perfect-gas, whilst it is usually supposed that the least luminous should first so deviate. Lastly, the proportionality (53) affords no explanation of the correlation of L with M observed in nature, for it permits any L to be associated with any M, provided L is not so large as to cause a deviation from the gas-laws (owing to r_1 coming out so small).

* As developed by Eddington, Jeans and Russell.

† More familiar in the form $L \propto T_c^{\frac{1}{2}} f(M)$, from which it follows immediately by combination with $L \propto T_c^4 r_1^2$. My paper, *Zeits. für Astrophys.*, **4**, 75, 1932, gives an independent proof of this formula for the boundary conditions $\rho = 0$, $T = 0$. I believe this to be the first rigorous proof yet given. The single valuedness of $f(M)$ has, however, not been established.

‡ *M.N.*, **91**, 109, 1930.

§ It is possible that the theory of gaseous configurations developed by the author in *Zeits. für Astrophys.*, **4**, 75, 1932, may ultimately yield more than one configuration satisfying the boundary conditions of nature, for prescribed L and M.

These results may be contrasted with the radius-behaviour predicted for the two-phase configurations of the standard model as now developed. Though in this paper I have constructed the theory only for κ_1, κ_2 = constant, the same properties may be expected to persist for opacity-laws such as $\kappa_1 \propto T^m\rho^n$ and $\kappa_2 \propto T^{m'}\rho^{n'}$, for the only effect of modifying the opacity-laws is to distort the families of curves of fig. 2 or fig. 3 without altering their general topography.

27. *Summary of the Properties of the Two-phase Configurations of the Generalised Standard Model.*—For those arbitrary values of L and M for which configurations can be constructed, the system exhibits (1) the existence of *pairs* of configurations, of two different radii, for sufficiently large L and M ($M > \mathfrak{M}$); (2) the existence of a *single* configuration for smaller L for large M ($M > \mathfrak{M}$); (3) the existence of a *single* configuration for smaller M ($M < \mathfrak{M}$) and any L not too large; (4) a nova phenomenon when the core-radius increases discontinuously after diminishing to zero; (5) an approximate mass-luminosity correlation over a considerable range of the possible configurations; (6) a pseudo-Cepheid phenomenon for configurations of the maximum luminosity for any given $M > \mathfrak{M}$; (7) the possibility of a variety of physical characteristics for configurations of large M, large L and large external radius.

In establishing these properties no numerical values of physical constants or observed M's and L's have been employed save in calculating the value of the critical mass $\mathfrak{M} = 2{\cdot}61\odot(2{\cdot}2\mu_{\text{H}}/\mu)^2$. This mass divides masses capable of exhibiting the nova and Cepheid phenomena from those incapable of exhibiting it. It also divides masses capable of existing in two configurations for the same L from those capable of existing in only one configuration for given L. The above properties are qualitatively possessed by the stars of nature, as we have seen.

28. *Absolute Value of the Radius.*—From A.S.S., I., p. 50, equation (116), or from equation (31) above, it is seen that the numerical value of the radius of any two-phase configuration is given by

$$r_1 = 2{\cdot}13 \times 10^8 \frac{\xi_0 f(a)}{\beta_1^{\frac{1}{3}}(1-\beta_1)^{\frac{1}{3}}}\left(\frac{\mu}{2\mu_H}\right)^{\frac{2}{3}} \text{cms.}$$

This gives large radii for β_1 small, *i.e.* for configurations of large L and M; and very large radii indeed for that one of the two solutions, for β_1 just greater than $\beta_1{}^0$, for which $a \sim 0$, since $f(a)$ becomes large too. Thus qualitatively the giant character is exhibited for certain configurations of large M and L. I have been unable to ascertain the complicated limit *

$$\lim_{a\to 0,\ \omega_3\to\omega_3{}^0} f(a;\ \omega_3)$$

for constant M, so that I am unable to state in explicit analytical form the limiting value of the radius of a giant configuration. But the indications are that the calculated radii of all configurations are much smaller than the

* This is a " single-limit problem." It determines the radius of a configuration as we pursue a mass-curve down to the axis of β_1.

observed radii of the stars they qualitatively resemble. For example, for $M = \mathfrak{M}$ we should expect that the value $\beta_1 \doteqdot \frac{4}{5}$, with the corresponding value of a very near zero, would give a configuration " resembling " in radius the A-stars of nature; whereas $\xi_0 f(a)\beta_1^{-\frac{1}{3}}(1-\beta_1)^{-\frac{1}{6}}$ appears not to yield a factor big enough to produce the observed radius of a typical A-star of mass $2{\cdot}6\odot$.

Thus the two-phase configurations of the generalised standard model, as investigated in this paper, are much smaller in external radius than the observed " corresponding " stars—save of course when the gaseous fringe is small, when the calculated radii reproduce very closely the observed radii of the white dwarfs and probably the radii of the O-type nuclei of planetary nebulæ.

It is possible that this is connected with the inappropriateness of the assumed boundary condition, $T=0$, $\rho=0$. The result of imposing these boundary conditions is that *only one* characteristic length emerges in the analysis, namely $(\frac{1}{3}a)^{\frac{1}{6}}K(\pi G)^{-\frac{1}{2}}(R/\mu)^{-\frac{2}{3}}$, the characteristic length defining the order of magnitude of the size of the white dwarfs. I have, however, carried out an investigation of both one-phase and two-phase configurations, taking account of boundary conditions.* A new characteristic length l appears, namely :

$$l=\left(\frac{c\sqrt{3}}{L}\right)^{\frac{1}{2}}\frac{(R/\mu)^2}{(4\pi)^{\frac{1}{2}}(\frac{1}{3}a)^{\frac{1}{2}}G}$$

$$=\left(\frac{L\odot}{L}\right)^{\frac{1}{6}}\left(\frac{L\mu_H}{\mu}\right)^2\times 5{\cdot}3\times 10^{17}\text{ cms.}$$

and also a second characteristic length, of the same order of magnitude, depending on the numerical coefficient in the law of opacity adopted for the gas-phase. These investigations strongly suggest, though I have not yet established the result, that there may exist complete gaseous configurations on the one hand, and gaseous-zone envelopes in two-phase configurations on the other hand, considerably bigger than are implied by the boundary condition, $T=0$, $\rho=0$. The endeavour to satisfy the Hopf-Bronstein boundary condition appears to stretch out the whole configuration. For purely gaseous configurations the new fundamental lengths suggest the existence of systems of radii comparable with the present *separations* of the stars, thus suggesting the former origin of our present stars as tenuous gas-spheres almost in contact. It is beyond the scope of the present discussion to treat of stellar evolution ; but I cannot refrain from hazarding the conjecture that our present stars, soon after segregation into these enormous gas-spheres, shrank cataclysmically, forming degenerate nuclei surrounded by gaseous envelopes whose size is a compromise between the large characteristic length l and the small characteristic length

$$(\tfrac{1}{3}a)^{\frac{1}{6}}K(\pi G)^{-\frac{1}{2}}(R/\mu)^{-\frac{2}{3}}.$$

The one length characterises certain gas-spheres, the other degenerate spheres, and composite configurations will have an intermediate size.

* *Zeits. für Astrophys.*, **4**, 75, 1932.

It is particularly to be noted that Eddington's theory of gaseous configurations, as it stands, predicts, for given M and L, a radius larger than the radius of the corresponding star in nature, unless a large proportion of hydrogen is assumed. Thus, as it were, the change-over from one-phase configurations to two-phase configurations of the same M and L over-corrects the calculated radius, as compared with the observed radius, from too large a value to too small a value. What used to be called an " opacity discrepancy " is better regarded as a " radius-discrepancy." We do not actually *observe* the interior opacity of a star; we actually observe L, M and r_1 (or L, M and T_e), and in all scientific comparisons of theory and observation it is generally admitted to be better to compare the values of actually observed variables with the predicted values than to compare theory and observation at some variable which we do not observe. Critics of the theory of two-phase configurations have usually concluded that the introduction of centrally-condensed models *increases* the opacity discrepancy. Actually it *reverses the sign* of the radius discrepancy, predicting a smaller radius than that observed (for given M and L), whereas the gas-theory (apart from the hydrogen hypothesis) predicts a larger radius. According to B. Strömgren (*Zeits. für Astrophs.*, 4, 148, 1932) the hydrogen hypothesis suffers from the disadvantage that different types of stars imply varying amounts of hydrogen, so that the " hydrogen-abundance " is simply an empirical parameter introduced in order to yield certain aspects of the Russell diagram. In this paper, on the contrary, the radius-characteristics of the Russell diagram have been predicted qualitatively without the introduction of a single observed quantity : gM stars, B, O stars, planetary nuclei, main-sequence stars, Cepheids, novæ and the mass-luminosity correlation are all implied by our fundamental diagram as far as *relative* radius values are concerned.

For these reasons I attach much more importance to the *qualitative* core-radius properties of two-phase configurations disclosed in this paper than to the quantitative value of the external radius. The core-radius properties will be probably insensitive to the boundary conditions, whilst the external radius is very sensitive to the boundary conditions. We have seen that the core-radius properties disclose a set of configurations with many of the observed characteristics of the stars. There is no reason to suspect that these will not persist when other source-distributions and other laws of opacity * are introduced, such as $\kappa_1 \propto T^m \rho^n$, $\kappa_2 \propto T^{m'} \rho^{n'}$. The fundamental fig. 3 with its families of curves will be distorted, but its topography will remain fundamentally the same. For this reason it has appeared worth while to undertake this intensive study of the generalised standard model. The insight it has now given into the observed qualitative characteristics of the stars justifies the labour hitherto spent.

It will be seen that the complete programme of identifying configurations has not been fully carried out, for we have not reproduced the radii of observed stars. We have traced general resemblances between the relative radii of the constructed configurations and those of the observed stars. I am quite

* I have carried out a great many investigations for various source-distributions and various opacity laws, and always find the same characteristic lengths emerging.

prepared to find, when further investigations have been carried out, that I have been unduly optimistic in my resemblances and identifications. I attach less importance to these identifications than to the fundamental principles developed and the spade-work of putting them into practice in a particular instance. The core of the paper is not the identifications but the construction of two-phase configurations of prescribed M and L and the determination of their properties.

As soon as the existence of two-phase configurations has been recognised it becomes the business of the investigator to enumerate their properties without any *a priori* considerations arising out of his own ideas as to the nature of the stars. The identifications then occur in an unforced way. If the only two-phase configurations to be found in nature occurred amongst the white dwarfs, we should at least have learned the properties to be expected amongst the white dwarfs and similar dense stars. Actually the constructed two-phase configurations of this paper reproduce not the properties of the white dwarfs alone but those of the main categories of the observed concourse of stars. A set of two-phase configurations erected in the heavens with all those arbitrary values of L and M for which configurations can be constructed would be classified by an observer in much the same way as the observed stars have been classified. The only trouble would be that the observed radii would be larger than those here calculated from the boundary conditions, $T=0$ and $\rho=0$. The only outstanding question is in fact the absolute value of the thickness of the gaseous envelope.

Investigations of the configurations of rotating incompressible masses do not lose their value because the heavenly bodies are not formed of incompressible matter ; these investigations afford a norm of behaviour, a background against which investigations of compressible masses can be projected ; they explore the way. Similarly I hope that my investigations of the generalised standard model may afford a norm of behaviour for comparison with other, more general, constructed systems and at the same time yield some insight into the nature of the observed stars.

I desire to express my thanks to the Department of Scientific and Industrial Research for a grant for " the investigation of spherically symmetrical aggregates of matter in equilibrium under their own gravitation for comparison with observed characteristics of the stars," and to thank also Mr. Fairclough for carrying out the integrations of Emden's equation, $n=\frac{5}{2}$.

Note added 1932 *May* 24.—The material of this paper is sufficient to enable the actual loci of r_1 against L or β_1 for the generalised standard model to be plotted in full. These promise to be highly interesting, as they exhibit the relationship of the Emden-Eddington polytropes to the two-phase configurations. But as a certain unsolved point has so far eluded me, and as a great deal of arithmetic is required to construct this diagram accurately, I defer it to another occasion. The reader may easily construct it for himself in broad outline.

APPENDIX

Tables of Solutions of Equations of Fit *

I. *Centrally-condensed Configurations*

TABLE I

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=1{\cdot}009139$ (centrally-condensed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·25	2·8009	0·4233	1·1584	7·92	3·2	0·234
·20	3·5711	·4980	1·0244	6·88	3·12	·238
·15	4·6988	·5577	·8908	6·10	3·06	·234
·12	...	·57	·80	6·0	3·05	·213
·10	6·5512	·5641	·7609	6·05	3·06	·201
·08	7·7203	·5345	·7137	6·38	3·08	·179
·06	9·4209	·4739	·6742	7·20	3·15	·1496
0	∞	0	1	∞	3·6538	0

TABLE II

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=1{\cdot}500$ (centrally-condensed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·40	2·1537	·4933	1·5044	6·93	3·12	0·347
·35	2·6000	·6266	1·3216	5·63	3·02	·374
·30	3·1431	·7768	1·1446	4·78	2·92	·383
·25	3·8091	·9357	·9691	3·92	2·79	·395
·225	4·1988	1·0125	·8815	3·58	2·72	·395
·20	4·6338	1·0814	·7942	3·35	2·66	·379
·175	5·1219	1·1348	·7080	3·18	2·63	·356
·15	5·6746	1·1604	·6243	3·11	2·62	·321
·125	6·3113	1·1404	·5457	3·17	2·63	·276
·10	7·0700	1·0493	·4764	3·46	2·70	·220
·08	7·8212	·9051	·4325	4·05	2·81	·171
·06	8·8180	·6884	·4066	5·25	2·98	·124
·04	10·400	·4175	·4144	8·0 ?	3·2 ?	·083 ?
0	∞	0	1·0	∞	3·6538	0

* From his previous tabulation of solutions of Emden's equation " $n=3$," values of u and v were calculated by Mr. Fairclough. The determination of solutions of the equations of fit was carried out by myself with the assistance of my wife. These tables and corresponding graphs can be used for the calculation of radii of two-phase configurations for any L, M, κ_1, κ_2.

I. *Centrally-condensed Configurations*—continued

TABLE III

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=1{\cdot}90$ (centrally-condensed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·50	1·8503	·4590	1·8647	7·43	3·16	0·401
·40	2·658	·8012	1·4110	4·61	2·88	·490
·35	3·1630	1·0214	1·1999	3·55	2·72	·541
·30	3·7432	1·2725	·9910	2·78	2·52	·570
·25	4·3974	1·5444	·7826	2·11	2·28	·595
·20	5·1129	1·8102	·5777	1·61	2·06	·577
·15	5·8627	2·0013	·3864	1·30	1·88	·476
·10	6·6175	1·9131	·2289	1·45	1·96	·253
·08	6·9255	1·6801	·1827	1·85	2·16	·158
·06	7·2625	1·2482	·1521	2·83	2·52	·086
·05	...	·94	·145	3·89	2·78	·0597
·04	7·7077	·6454	·1473	5·53	3·01	·0426
·03	...	·40	·155	8·4 ?	3·21	·0295
0	∞	0	1	∞	3·6538	0

II. *Quasi-diffuse Configurations*

TABLE IV

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=2{\cdot}018$ (Emden's solution) $\xi_0=6{\cdot}897$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
0	6·897	3	0	0		0·8
·25	6·826	2·964	·0207	0·045	...	...
·5	6·620	2·856	·0807	0·155	0·67	...
·75	6·300	2·690	·1745	0·36	1·02	·775
1·0	5·897	2·479	·2950	·625	1·34	·754
1·2	5·535	2·291	·4048	·880	1·56	·737
1·4	5·155	2·095	·5227	1·155	1·78	·728
1·6	4·769	1·897	·6455	1·455	1·96	·711
1·8	4·388	1·701	·7710	1·815	2·15	·680
2·0	4·020	1·514	·8974	2·18	2·30	·658
2·25	3·584	1·294	1·0558	2·73	2·5	·618
2·5	3·180	1·094	1·2144	3·31	2·66	·586
2·75	2·811	·915	1·3742	4·02	2·81	·547
3·0	2·477	·755	1·5372	4·91	2·94	·500
3·25	2·176	·616	1·7064	5·70	3·02	·479
3·5	1·905	·496	1·8859	6·86	3·12	·440
6·8969	0	0	∞	∞	3·6538	0

III. *Collapsed Configurations*

TABLE V

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=2{\cdot}11920$ (collapsed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u=(U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·5425	1·75	·4371	2·0621	7·65	3·18	·432
·5068	2·0	·5501	1·8677	6·23	3·07	·478
·4683	2·3	·6932	1·6735	5·22	2·97	·512
·4335	2·6	·8429	1·5072	4·39	2·86	·548
·4017	2·9	·9981	1·3596	3·63	2·74	·599
·3722	3·2	1·1582	1·2248	3·11	2·62	·630
·3446	3·5	1·3230	1·0993	2·63	2·48	·667
·3183	3·8	1·4927	·9803	2·23	2·34	·703
·2932	4·1	1·6680	·8662	1·87	2·18	·740
·2687	4·4	1·8502	·7554	1·545	2·02	·782
·2445	4·7	2·0421	·6468	1·225	1·84	·841
·2203	5·0	2·2486	·5394	0·945	1·64	·913
·1953	5·3	2·4812	·4318	0·62	1·33	1·116
·1686	5·6	2·7697	·3219	0·26	·89	1·980
·153	...	3·0	·25	0·0	0	∞

TABLE VI

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=4{\cdot}03626$ (collapsed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·70	1·7119	·4455	3·2237	7·60	3·18	·678
·65	2·1281	·7149	2·6763	5·14	2·96	·832
·60	2·5871	1·0851	2·2204	3·32	2·66	1·069
·55	3·0816	1·5855	1·8119	2·015	2·24	1·440
·50	3·5940	2·2677	1·4240	·92	1·60	2·480
·463	...	3·0	1·11	0	0	∞

III. *Collapsed Configurations*—continued

TABLE VII

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=8{\cdot}073108$ (collapsed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
0·8	2·004	·5268	4·8800	6·50	3·08	1·202
·75	2·642	1·0449	3·7579	3·48	2·70	1·728
·7	3·320	1·8604	2·9035	1·53	2·01	3·04
·6735	...	2·5	2·45	·60	1·31	6·53
·6547	...	3·0	2·215	0	0	∞

TABLE VIII

$(n=\frac{3}{2})$ $\omega_{\frac{3}{2}}=132{\cdot}39$ (Emden's solution) $\eta_0=3{\cdot}6538$
$(n=3)$ $\omega_3=20{\cdot}18277$ (collapsed solution) $\xi_0=1$

a	$\xi_0\theta(a)$	$u(=U)$	v	V	b(rough)	$\frac{\beta_1}{\beta_2}=\frac{8}{5}\frac{v}{V}$
1·0	0	0	∞	∞	3·6538	0
·85	3·5167	1·4042	6·3635	2·435	2·42	4·18
·81	...	3·0	4·63	0	0	∞

TABLES OF SOLUTIONS OF $\psi''+\frac{2}{\eta}\psi'=-\psi^{\frac{3}{2}}$ SATISFYING THE INITIAL CONDITIONS, $-\eta_0^{5}\psi'(\eta_0)=\omega_{\frac{3}{2}}$ AND $\psi(\eta_0)=0$, WHERE η_0 AND $\omega_{\frac{3}{2}}$ ARE PRESCRIBED.

N. Fairclough.

(Communicated by E. A. Milne)

If $\psi(\eta)$ is a solution of Emden's polytropic equation

$$\psi''+\frac{2}{\eta}\psi'=-\psi^n$$

we define numbers ω_n and η_0 by the relations

$$\omega_n=-\eta_0^{\frac{n+1}{n-1}}\psi'(\eta_0) \qquad \text{and} \qquad \psi(\eta_0)=0.$$

Emden's tabulated solutions of this equation, found by outward integration from $\eta=0$ onwards, correspond to $\psi(0)=1$, $\psi'(0)=0$, and lead to a value η_0 for which $\psi=0$. From this value of η_0 we can calculate $\omega_n{}^0$, the value of ω_n in this special case. Integrations may then be carried out inwards from the point $\eta=\eta_0$, with ω_n taking any prescribed value. It is known that when

$$\omega_n<\omega_n{}^0, \qquad \psi(\eta)\to\infty \text{ as } \eta\to 0,$$

and when

$$\omega_n>\omega_n{}^0,$$

$\psi(\eta)$ rises to a maximum and decreases to zero before η vanishes.

Professor Milne's investigations of stellar structure require the consideration of associated functions U and V, defined by

$$U=-\frac{\eta\psi^n}{\psi'}, \qquad V=-\frac{\eta\psi'}{\psi}.$$

It is convenient to define also functions z and w by the relations

$$z=\eta^{\frac{2}{n-1}}\psi(\eta), \qquad w=-\eta^{\frac{n+1}{n-1}}\psi'(\eta),$$

so that

$$U=\frac{z^n}{w}, \qquad V=\frac{w}{z}.$$

If $\psi=f(\eta)$ is any solution of $\psi''+\frac{2}{\eta}\psi'=-\psi^n$, then it can be shown that $\psi=A^{\frac{2}{n-1}}f(A\eta)$ is also a solution, and the derived functions (z, w), (U, V) are the same functions of one another whatever the choice of A.* Also, ω_n is independent of the chosen value of A.

* I owe this observation to Professor Milne.

In this paper tables are communicated for $n=\frac{3}{2}$. Tables for $n=3$ have already been published (*M.N.*, **91**, 55, 1930). These tables have been constructed by reducing the system to two first-order equations and using the method described on page 546 of Ince's *Ordinary Differential Equations* (1927). To secure uniformity, Emden's solution was recalculated and gave

$$\eta_0 = 3{\cdot}65379: \qquad \omega_{\frac{3}{2}}{}^0 = 132{\cdot}3885.$$

The inward integrations were carried out with $\eta_0 = 3{\cdot}65379$ and the following values of $\omega_{\frac{3}{2}} = -\eta_0{}^5\psi'(\eta_0)$:—

$\omega_{\frac{3}{2}}(< \omega_{\frac{3}{2}}{}^0) = 105{\cdot}9108$; 66·19422 ; 33·09711.
$\omega_{\frac{3}{2}}(> \omega_{\frac{3}{2}}{}^0) = 529{\cdot}5538$; 264·7769.

These six tables are given below.

TABLE I

(*Emden's Solution:* $n=\frac{3}{2}$)

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}}{}^0 = 132{\cdot}3885$

η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	$V\left(=-\frac{\eta\psi'}{\psi}\right)$
0	1	0	0	0	3	0
·027369	·998335	·0332834	$·998335\times10^{-4}$	$·332834\times10^{-6}$	2·99700	$·333389\times10^{-2}$
·082107	·985101	·0986602	$·797932\times10^{-2}$	$·239744\times10^{-3}$	2·97304	·0300457
·136844	·959104	·160545	·0599440	$·501703\times10^{-2}$	2·92531	·0836952
·164213	·941587	·189515	·122030	·0147367	2·89266	·120763
·191582	·921253	·216860	·221193	·0364477	2·85421	·164778
·232635	·885850	·254389	·462419	·112874	2·78587	·244094
·273688	·845163	·287246	·845163	·287246	2·70493	·339871
·328426	·783966	·323096	1·62563	·803967	2·57808	·494556
·383164	·716552	·349401	2·75271	1·87916	2·43039	·682660
·437901	·644840	·366123	4·22602	3·83908	2·26293	·908438
·492639	·570713	·373670	5·99112	7·06075	2·07688	1·178536
·547377	·495932	·372820	7·93492	11·93025	1·87355	1·503513
·602114	·422075	·364632	9·88736	18·7918	1·65444	1·900589
·656852	·350487	·350348	11·62832	27·8969	1·42141	2·399045
·711590	·282255	·331300	12·89837	39·3630	1·17683	3·051783
·766327	·218196	·308836	13·41155	53·1518	·924062	3·963133
·821065	·158863	·284250	12·86791	69·0727	·668276	5·367824
·875803	·104558	·258749	10·96370	86·8217	·418126	7·91902
·930540	·0553514	·233441	7·39681	106·0651	·189668	14·33931
·985278	·0110987	·209393	1·86416	126·6117	·0201025	67·9191
6 ·999948	$·3892\times10^{-4}$	·203319	·0069316	132·3677	$·43632\times10^{-5}$	19096·3
79 1	0	·203297	0	132·3885	0	∞

TABLE II

(*Centrally-condensed Solution*) $(\omega_{\frac{3}{2}} < \omega_{\frac{3}{2}}{}^{0})$

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}} = 105{\cdot}91077$

η	η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	V $\left(-\right.$
3·65379	1	0	·162638	0	105·91077	0	∞
3·65	·998963	$\cdot61720 \times 10^{-3}$	·162976	·109546	105·5814	$\cdot343407 \times 10^{-3}$	963·
3·63	·993489	$\cdot389466 \times 10^{-2}$	·164774	·676233	103·8539	$\cdot535454 \times 10^{-2}$	153·
3·6	·985278	$\cdot887890 \times 10^{-2}$	·167516	1·491314	101·2907	$\cdot179798 \times 10^{-1}$	67·9
3·5	·957909	$\cdot261000 \times 10^{-1}$	·176983	3·91663	92·9550	$\cdot833865 \times 10^{-1}$	23·7
3·4	·930540	$\cdot442894 \times 10^{-1}$	·186867	5·91855	84·9037	·169589	14·3
3·2	·875803	$\cdot837124 \times 10^{-1}$	·207529	8·77788	69·6351	·373471	7·9
3·0	·821065	·127344	·228844	10·31485	55·6092	·595728	5·3
2·8	·766327	·175249	·250147	10·77176	43·0512	·821192	3·9
2·6	·711590	·227353	·270725	10·38950	32·1658	1·041113	3·0
2·4	·656852	·283439	·289833	9·40383	23·0783	1·249548	2·45
2·2	·602114	·343138	·306736	8·03822	15·8080	1·441655	1·96
2·0	·547377	·405941	·320774	6·49505	10·26476	1·612593	1·58
1·8	·492639	·471223	·331480	4·94671	6·26354	1·756527	1·26
1·6	·437901	·538302	·338767	3·52782	3·55223	1·865339	1·00
1·4	·383164	·606541	·343262	2·33009	1·84614	1·926605	·79
1·2	·328426	·675550	·346981	1·40082	·863399	1·920271	·61
1·0	·273688	·745606	·354947	·745606	·354947	1·813843	·47
·8	·218951	·818621	·379811	·335307	·124456	1·560082	·37
·6	·164213	·900980	·458647	·116767	$\cdot356644 \times 10^{-1}$	1·118782	·30
·4	·109475	1·014657	·740603	$\cdot259752 \times 10^{-1}$	$\cdot758377 \times 10^{-2}$	·552019	·29
·3	$\cdot821065 \times 10^{-1}$	1·106916	1·168529	$\cdot896602 \times 10^{-2}$	$\cdot283953 \times 10^{-2}$	·298988	·31
·2	$\cdot547377 \times 10^{-1}$	1·273975	2·42882	$\cdot203836 \times 10^{-2}$	$\cdot777224 \times 10^{-3}$	·118406	·38
·1	$\cdot273688 \times 10^{-1}$	1·746365	9·32350	$\cdot174637 \times 10^{-3}$	$\cdot932350 \times 10^{-4}$	$\cdot247527 \times 10^{-1}$	·53
·05	$\cdot136844 \times 10^{-1}$	2·673106	36·9595	$\cdot167069 \times 10^{-4}$	$\cdot115499 \times 10^{-4}$	$\cdot591246 \times 10^{-2}$	·69
·01	$\cdot273688 \times 10^{-2}$	10·04685	921·197	$\cdot100469 \times 10^{-6}$	$\cdot921197 \times 10^{-7}$	$\cdot345694 \times 10^{-3}$	·91
·005	$\cdot136844 \times 10^{-2}$	19·25795	3684·25	$\cdot120362 \times 10^{-7}$	$\cdot115133 \times 10^{-7}$	$\cdot114693 \times 10^{-3}$	·95
·001	$\cdot273688 \times 10^{-3}$	92·9391	92100·1	$\cdot929391 \times 10^{-10}$	$\cdot921001 \times 10^{-10}$	$\cdot972831 \times 10^{-5}$	·99
·0005	$\cdot136844 \times 10^{-3}$	185·039	368399	$\cdot115649 \times 10^{-10}$	$\cdot115125 \times 10^{-10}$	$\cdot341622 \times 10^{-5}$	·99
·0001	$\cdot273688 \times 10^{-4}$	921·835	9209951	$\cdot921835 \times 10^{-13}$	$\cdot920995 \times 10^{-13}$	$\cdot303893 \times 10^{-6}$	·99

Table III

(*Centrally-condensed Solution*) ($\omega_{\frac{3}{2}} < \omega_{\frac{3}{2}}{}^{0}$)

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}} = 66{\cdot}19422$

η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	$V=\left(-\frac{\eta\psi'}{\psi}\right)$
1	0	·101649	0	66·19422	0	∞
·999948	·19416 $\times 10^{-4}$	·101659	·345974 $\times 10^{-2}$	66·1838	·307478 $\times 10^{-5}$	19129·7
·998963	·38575 $\times 10^{-3}$	·101860	·684664 $\times 10^{-1}$	65·9884	·271487 $\times 10^{-3}$	963·807
·993489	·243416 $\times 10^{-2}$	·102984	·422645	64·9089	·423312 $\times 10^{-2}$	153·578
·985278	·554934 $\times 10^{-2}$	·104700	·932077	63·3081	·142141 $\times 10^{-1}$	67·9216
·971504	·108573 $\times 10^{-1}$	·107632	1·724386	60·6854	·373137 $\times 10^{-1}$	35·1924
·957909	·163140 $\times 10^{-1}$	·110649	2·44811	58·1148	·659114 $\times 10^{-1}$	23·7386
·944225	·219235 $\times 10^{-1}$	·113745	3·10589	55·5940	·984580 $\times 10^{-1}$	17·8995
·930540	·276897 $\times 10^{-1}$	·116917	3·70028	53·1217	·133992	14·3561
·903171	·397070 $\times 10^{-1}$	·123473	4·70893	48·3217	·211466	10·26173
·875803	·523932 $\times 10^{-1}$	·130293	5·49382	43·7191	·294537	7·95787
·848434	·657736 $\times 10^{-1}$	·137354	6·07433	39·3234	·380713	6·47369
·821065	·798713 $\times 10^{-1}$	·144636	6·46958	35·1465	·468200	5·43258
·766327	·110301	·159782	6·77970	27·4991	·641944	4·05610
·711590	·143828	·175593	6·57259	20·8629	·807665	3·17422
·656852	·180575	·191971	5·99104	15·2859	·959318	2·55146
·602114	·220653	·208909	5·16892	10·7664	1·091514	2·08291
·547377	·264185	·226575	4·22697	7·25039	1·198620	1·71527
·492639	·311362	·245465	3·26855	4·63822	1·274034	1·41905
·437901	·362523	·266690	2·37583	2·79645	1·311345	1·17704
·383164	·418341	·292559	1·60710	1·57345	1·294823	·979063
·328426	·480163	·327856	·995665	·815811	1·217812	·819363
·273688	·550780	·383048	·550780	·383048	1·067122	·695464
·218951	·636320	·483622	·260637	·158473	·839648	·608024
·164213	·751892	·704245	·974452 $\times 10^{-1}$	·547621 $\times 10^{-1}$	·555471	·561978
·109475	·944094	1·34685	·241688 $\times 10^{-1}$	·137918 $\times 10^{-1}$	·272436	·570642
·821065 $\times 10^{-1}$	1·117665	2·25409	·905309 $\times 10^{-2}$	·547743 $\times 10^{-2}$	·157260	·605035
·547377 $\times 10^{-1}$	1·447291	4·85260	·231567 $\times 10^{-2}$	·155283 $\times 10^{-2}$	·717612 $\times 10^{-1}$	·670576
·273688 $\times 10^{-1}$	2·400253	18·8797	·240025 $\times 10^{-3}$	·188797 $\times 10^{-3}$	·196965 $\times 10^{-1}$	·786573
·136844 $\times 10^{-1}$	4·27854	74·9261	·267409 $\times 10^{-4}$	·234144 $\times 10^{-4}$	·590583 $\times 10^{-2}$	·875603
·273688 $\times 10^{-2}$	19·22376	1866·885	·192238 $\times 10^{-6}$	·186689 $\times 10^{-6}$	·451482 $\times 10^{-3}$	·971134
·136844 $\times 10^{-2}$	37·8903	7466·10	·236814 $\times 10^{-7}$	·233316 $\times 10^{-7}$	·156195 $\times 10^{-3}$	·985226
·273688 $\times 10^{-3}$	187·201	186635	·187201 $\times 10^{-9}$	·186635 $\times 10^{-9}$	·137237 $\times 10^{-4}$	·996975
·136844 $\times 10^{-3}$	373·836	746536	·233647 $\times 10^{-10}$	·233292 $\times 10^{-10}$	·484106 $\times 10^{-5}$	·998481
·273688 $\times 10^{-4}$	1866·90	18663330	·186690 $\times 10^{-12}$	·186633 $\times 10^{-12}$	·432208 $\times 10^{-6}$	·999695

Table IV

(Centrally-condensed Solution) $(\omega_{\frac{3}{2}} < \omega_{\frac{3}{2}}{}^{0})$

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}} = 33{\cdot}09711$

η	η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	V $\left(-\right.$
3·65379	1	0	·0508243	0	33·09711	0	∞
3·65	·998963	·192875 × 10^{-3}	·0509299	·342332 × 10^{-1}	32·9942	·191970 × 10^{-3}	963·
3·63	·993489	·121708 × 10^{-2}	·0514923	·211323	32·4545	·299326 × 10^{-2}	153·
3·6	·985278	·277469 × 10^{-2}	·0523513	·466042	31·6548	·100507 × 10^{-1}	67·9
3·5	·957909	·815780 × 10^{-2}	·0553433	1·224179	29·0673	·465975 × 10^{-1}	23·7
3·4	·930540	·138498 × 10^{-1}	·0585277	1·85079	26·5923	·946849 × 10^{-1}	14·3
3·2	·875803	·262368 × 10^{-1}	·0654721	2·75113	21·9688	·207711	7·9
3·0	·821065	·400914 × 10^{-1}	·0732116	3·24741	17·7904	·328941	5·47
2·8	·766327	·555787 × 10^{-1}	·0818137	3·41624	14·0804	·448442	4·12
2·6	·711590	·728818 × 10^{-1}	·0913945	3·33052	10·8589	·559734	3·26
2·4	·656852	·922139 × 10^{-1}	·102144	3·05944	8·13333	·657951	2·65
2·2	·602114	·113837	·114371	2·66670	5·89425	·738811	2·21
2·0	·547377	·138094	·128585	2·20950	4·11471	·798181	1·86
1·8	·492639	·165459	·145644	1·73692	2·75204	·831794	1·58
1·6	·437901	·196637	·167040	1·28868	1·75154	·835213	1·35
1·4	·383164	·232739	·195485	·894088	1·05136	·804113	1·17
1·2	·328426	·275638	·236224	·571562	·587802	·735131	1·02
1·0	·273688	·328756	·300343	·328756	·300343	·627614	·91
·8	·218951	·399016	·414454	·163437	·135808	·486518	·83
·6	·164213	·502687	·656131	·651482 × 10^{-1}	·510208 × 10^{-1}	·325917	·78
·4	·109475	·688722	1·338667	·176313 × 10^{-1}	·137080 × 10^{-1}	·170786	·77
·3	·821065 × 10^{-1}	·863342	2·288904	·699307 × 10^{-2}	·556204 × 10^{-2}	·105140	·79
·2	·547377 × 10^{-1}	1·20051	4·99395	·192081 × 10^{-2}	·159806 × 10^{-2}	·526786 × 10^{-1}	·83
·1	·273688 × 10^{-1}	2·18544	19·5500	·218544 × 10^{-3}	·195500 × 10^{-3}	·165257 × 10^{-1}	·89
·05	·136844 × 10^{-1}	4·13178	77·6643	·258236 × 10^{-4}	·242701 × 10^{-4}	·540698 × 10^{-2}	·93
·01	·273688 × 10^{-2}	19·6256	1935·45	·196256 × 10^{-6}	·193545 × 10^{-6}	·449213 × 10^{-3}	·98
·005	·136844 × 10^{-2}	38·9777	7740·31	·243611 × 10^{-7}	·241885 × 10^{-7}	·157194 × 10^{-3}	·99
·001	·273688 × 10^{-3}	193·772	193489	·193772 × 10^{-9}	·193489 × 10^{-9}	·139406 × 10^{-4}	·99
·0005	·136844 × 10^{-3}	387·261	773953	·242038 × 10^{-10}	·241860 × 10^{-10}	·492335 × 10^{-5}	·99
·0001	·273688 × 10^{-4}	1935·16	19348770	·193516 × 10^{-12}	·193488 × 10^{-12}	·439970 × 10^{-6}	·99

Table V

(*Collapsed Solution*) $(\omega_{\frac{3}{2}} > \omega_{\frac{3}{2}}{}^{0})$

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}} = 529{\cdot}5538$

	η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	$V\left(=-\frac{\eta\psi'}{\psi}\right)$
379	1	0	·8131889	0	529·5538	0	∞
	·998963	·308601 × 10^{-2}	·814879	·547733	527·907	·767884 × 10^{-3}	963·804
	·993489	·194732 × 10^{-1}	·823858	3·38115	519·260	·119732 × 10^{-1}	153·575
	·985278	·443928 × 10^{-1}	·837470	7·45628	506·386	·402070 × 10^{-1}	67·9141
	·957909	·130430	·883304	19·5726	463·928	·186648	23·7029
	·930540	·2210295	·928432	29·5370	421·837	·380544	14·2817
	·875803	·415139	1·010092	43·5305	338·931	·847383	7·70826
	·821065	·623585	1·069642	50·5104	259·923	1·381104	5·14593
	·766327	·840632	1·093885	51·6700	188·262	1·972856	3·64354
	·711590	1·057883	1·069469	48·3427	127·068	2·64522	2·62847
	·656852	1·264316	·983674	41·9470	78·3262	3·46851	1·86727
	·602114	1·446507	·825299	33·8853	42·5329	4·63759	1·25520
	·547377	1·588993	·585189	25·4239	18·7261	6·84568	·736554
	·520008	1·640048	·432101	21·3733	10·6992	9·23535	·500590
	·492639	1·674640	·255771	17·5797	4·83297	15·2512	·274918
	·465270	1·690393	·0551015	14·1183	·782363	67·8058	·0554147
456	·458308	1·691098	0	13·2976	0	∞	0
	·451586	1·690428	− ·0548081	12·5295	− ·670292	−66·1659	− ·0534973
	·437901	1·684801	− ·171418	11·0415	−1·79744	−20·4121	− ·162790
	·410533	1·655175	− ·426057	8·37933	−3·23537	− 7·49704	− ·386113
	·383164	1·598538	− ·712403	6·14094	−3·83147	− 3·97180	− ·623922
	·355795	1·511454	−1·036203	4·31687	−3·84735	− 2·33126	− ·891237
	·328426	1·389751	−1·406737	2·88179	−3·50041	− 1·39757	−1·21467
	·273688	1·018042	−2·35783	1·01804	−2·35783	− ·435648	−2·31605
	·218951	·410364	−3·85606	·168085	−1·26355	− ·0545381	−7·51735
5227	·193286	0	−4·96056	0	− ·871479	0	− ∞

TABLE VI

(*Collapsed Solution*) $(\omega_{\frac{3}{2}} > \omega_{\frac{3}{2}}{}^0)$

$\eta_0 = 3{\cdot}65379$; $\omega_{\frac{3}{2}} = 264{\cdot}7769$

η	η/η_0	ψ	$-\psi'$	$z(=\eta^4\psi)$	$w(=-\eta^5\psi')$	$U\left(=-\frac{\eta\psi^{\frac{3}{2}}}{\psi'}\right)$	$V($ =
3·65379	1	0	·4065945	0	264·7769	0	∞
3·6536	·999948	·77664 × 10⁻⁴	·406637	·138390 × 10⁻¹	264·735	·614956 × 10⁻⁵	19129·7
3·65	·998963	·154300 × 10⁻²	·407439	·273866	263·954	·542974 × 10⁻³	963·807
3·63	·993489	·973663 × 10⁻²	·411933	1·69058	259·633	·846630 × 10⁻²	153·576
3·6	·985278	·221968 × 10⁻¹	·418765	3·72822	253·211	·284295 × 10⁻¹	67·917
3·5	·957909	·652335 × 10⁻¹	·442079	9·78910	232·188	·131909	23·719
3·4	·930540	·110625	·465778	14·7833	211·628	·268585	14·3154
3·2	·875803	·208475	·512324	21·8602	171·908	·594547	7·863
3·0	·821065	·315257	·554437	25·5359	134·728	·957783	5·276
2·8	·766327	·429671	·587906	26·4100	101·181	1·341387	3·831
2·6	·711590	·549537	·608181	25·1125	72·2602	1·74155	2·8774
2·4	·656852	·671739	·610452	22·2867	48·6080	2·16452	2·1816
2·2	·602114	·792185	·589787	18·5574	30·3954	2·63007	1·6379
2·0	·547377	·905793	·541228	14·4927	17·3193	3·18561	1·1956
1·8	·492639	1·006483	·459713	10·56566	8·68660	3·95362	·8221
1·6	·437901	1·087115	·339547	7·12452	3·56040	5·34114	·4997
1·5	·410533	1·117328	·262668	5·65647	1·99464	6·74457	·3526
1·4	·383164	1·139219	·172837	4·37642	·929558	9·84923	·2124
1·3	·355795	1·151400	·680872 × 10⁻¹	3·28852	·252803	23·5894	·768746
1·2425	·340058	1·153388	·268492 × 10⁻⁴	2·74891	·795085 × 10⁻⁴	57322·9	·289236
1·24248	·340052	1·153388	0	2·74872	0	∞	0
1·2	·328426	1·152244	− ·0544753	2·38929	−·135552	−27·2457	− ·056
1·1	·301057	1·139768	− ·199246	1·66874	−·320887	− 6·71783	− ·192
1·0	·273688	1·111430	− ·373181	1·11143	−·373181	− 3·13981	− ·335
·8	·218951	·991970	− ·862130	·406311	−·282503	− ·916780	− ·695
·6	·164213	·740336	−1·76426	·959476 × 10⁻¹	−·137189	− ·216637	−1·429
·4	·109475	·198099	−4·09933	·507133 × 10⁻²	−·419771 × 10⁻¹	−·860341 × 10⁻²	−8·277
·358	·979804 × 10⁻¹	·567625 × 10⁻²	−5·11948	·932381 × 10⁻⁴	−·301051 × 10⁻¹	−·299054 × 10⁻⁴	−322·8
·3569	·976794 × 10⁻¹	·2747 × 10⁻⁴	−5·15108	·445702 × 10⁻⁶	−·298285 × 10⁻¹	−·997553 × 10⁻⁸	−66924
·356895	·976780 × 10⁻¹	0	−5·15123	0	−·298272 × 10⁻¹	0	− ∞

Conclusions from the Foregoing Tables

(i) $\omega_{\frac{3}{2}} < \omega_{\frac{3}{2}}{}^0$.—For these solutions for η small, $\psi \sim \frac{A}{\eta}$ where A is a constant depending on the value of $\omega_{\frac{3}{2}}$. The above tables give the following approximations for A:—

TABLE VII

$\omega_{\frac{3}{2}} = 33{\cdot}09711$, $A \sim 0{\cdot}19348$
$\omega_{\frac{3}{2}} = 66{\cdot}19422$, $A \sim 0{\cdot}18663$
$\omega_{\frac{3}{2}} = 105{\cdot}9108$, $A \sim 0{\cdot}092099$

(ii) $\omega_{\frac{3}{2}} > \omega_{\frac{3}{2}}{}^0$.—It is known that ψ has a first maximum and a first zero in the interval $\eta_0 > \eta > 0$ as we proceed inwards from the zero $\eta = \eta_0$. The

following short table gives the values of the maximum and the zero, and the corresponding values of η :—

TABLE VIII

	Maximum value of ψ	First zero of ψ
$\omega_{\frac{3}{2}} = 264{\cdot}7769$	$\psi = 1{\cdot}15339$ at $\eta = 1{\cdot}24248$	at $\eta = 0{\cdot}356895$
$\omega_{\frac{3}{2}} = 529{\cdot}5538$	$\psi = 1{\cdot}69110$ at $\eta = 1{\cdot}67456$	at $\eta = 0{\cdot}706227$

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THE MINIMUM PRESSURE OF A DEGENERATE ELECTRON GAS.

Edmund C. Stoner, Ph.D.

(Communicated by Sir A. S. Eddington)

A knowledge of the variation of pressure with density of matter at very high densities is essential in the investigation of stellar structure. Very high densities are known to occur in white dwarfs, and recent theoretical developments suggest that they may also occur in the central regions of more ordinary stars.* At these densities matter may be regarded as an aggregate of nuclei and electrons, the electrons being treated either as free, or as constituting, with the nuclei, a gigantic composite molecule. In either case, the material pressure is almost entirely due to the electrons, and calculations of the pressure may be made by treating the electrons as a gas subject to the Fermi-Dirac statistics. The pressure depends on the temperature, but for the range of electron concentrations under consideration, unless the temperature is very high, it does not differ greatly from the value at absolute zero. The pressures calculated for absolute zero temperature (corresponding to *complete* degeneracy of the electron gas) are, of course, minimum pressures. Provided that the electron concentration, n, is not too large, the pressure varies as $n^{5/3}$. This holds as long as the momenta are such that the relativity change of mass with velocity may be neglected. At higher concentrations, this relativity effect has to be taken into account. Eventually the pressure varies as $n^{4/3}$. Application has been made of the equations of state corresponding to these two limits,*† but concentrations which are of astrophysical interest (corresponding roughly to densities of the order 10^5 to 10^8) happen to fall in the range where neither equation can be strictly applied—possibly *because* this is a transition region. It has seemed worth while, therefore, to investigate how the pressure varies over the whole range from ordinary to

* E. A. Milne, *M.N.*, **91**, 4, 1930.
† S. Chandrasekhar, *ibid.*, **91**, 456, 1931.

relativistic degeneracy. In a previous paper * on the limiting density of white dwarfs this transition from ordinary to relativistic degeneracy was quantitatively discussed, but no explicit calculations of the pressures were made; though certain functions closely connected with the pressure were computed with a view to the particular application. Reference may be made to this paper for a fuller discussion of the astrophysical application, and to two other papers in the same series; † also to another paper by Chandrasekhar, ‡ as well as to the papers mentioned above. The present paper is concerned simply with the computation and the results for the minimum pressure and kinetic energy density of a degenerate electron gas; and an attempt is made to give the results in such a form that application of them can readily be made.

Derivation of Expressions for Pressure and Energy

The following symbols will be used :—

n = number of electrons per unit volume.
P = pressure (due to electrons).
E = kinetic energy of electrons per unit volume.
ϵ = kinetic energy of electron.
p = momentum of electron.
p_0 = maximum momentum.

In the completely degenerate state there are two electrons in each cell of the 6-dimensional phase space (of 6-volume h^3), 2 being the weight factor. The cells corresponding to the lowest possible energy will be filled, so that the total number of electrons, N, in a volume V is given by

$$N = (V/h^3)\int_0^{p_0} 2 \times 4\pi p^2 dp,$$

$$n = N/V = \frac{8\pi}{h^3}\int_0^{p_0} p^2 dp = \frac{8\pi {p_0}^3}{h^3}, \tag{1}$$

$$p_0 = \left(\frac{3h^3 n}{8\pi}\right)^{1/3}. \tag{1'}$$

Ultimately values for E and P in terms of n are required; the simplest procedure seems to be to obtain expressions for E and P first in terms of p, and then, by integration, in terms of p_0.

$$\epsilon = (m - m_0)c^2$$

$$= m_0 c^2\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right), \tag{2}$$

$$p^2 = (m\beta c)^2 = {m_0}^2 c^2\left(\frac{1}{1-\beta^2} - 1\right). \tag{3}$$

* E. C. Stoner, *Phil. Mag.*, 9, 944, 1930.

† E. C. Stoner, *ibid.*, 7, 63, 1929; E. C. Stoner and F. Tyler, *ibid.*, 11, 986, 1931.

‡ S. Chandrasekhar, *ibid.*, 11, 592, 1931.

The expressions are simplified by using x and y, defined by

$$\left.\begin{aligned} y &= \frac{p}{m_0 c} \\ x &= \frac{p_0}{m_0 c} \end{aligned}\right\}. \tag{4}$$

From (1′) and (4),

$$\left.\begin{aligned} x &= \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3} = \alpha n^{1/3} \\ n &= \left(\frac{m_0 c}{h}\right)^3 \left(\frac{8\pi}{3}\right) x^3 = \frac{1}{\alpha^3} x^3 \end{aligned}\right\}. \tag{5}$$

(3) may be written

$$y^2 = \frac{1}{1-\beta^2} - 1,$$

giving

$$\beta = \left(\frac{y^2}{1+y^2}\right)^{1/2}. \tag{6}$$

Substituting in (2),

$$\epsilon = m_0 c^2 \{(1+y^2)^{1/2} - 1\}. \tag{7}$$

The energy per unit volume,

$$E = \frac{8\pi}{h^3}\int \epsilon p^2 dp$$

$$= 8\pi\left(\frac{m_0 c}{h}\right)^3 m_0 c^2 \int_0^x y^2 \{(1+y^2)^{1/2} - 1\} dy.$$

Let

$$A = \frac{8\pi (m_0 c)^5}{3 m_0 h^3}. \tag{8}$$

Then

$$E = A \times 3 \int_0^x y^2 \{(1+y^2)^{1/2} - 1\} dy. \tag{9}$$

An expression will now be derived for the pressure. The ordinary transport of momentum result, $P = \frac{1}{3} \times v \times p \times n$, may be generalised to

$$P = \tfrac{1}{3}\int \beta c p dn$$

$$= \tfrac{1}{3}\int_0^{p_0} \beta c p \frac{8\pi p^2}{h^3} dp$$

$$= \frac{8\pi (m_0 c)^5}{3 m_0 h^3} \int_0^x y^4 (1+y^2)^{-1/2} dy$$

$$= A \int_0^x y^4 (1+y^2)^{-1/2} dy. \tag{10}$$

Evaluation of Integrals

$$E/A = f_E(x) = 3\int_0^x y^2(1+y^2)^{1/2}dy - x^3. \tag{11}$$

Since
$$\frac{d}{dy}\{y(1+y^2)^{3/2}\} = 3y^2(1+y^2)^{1/2} + (1+y^2)^{3/2}$$
$$= 4y^2(1+y^2)^{1/2} + (1+y^2)^{1/2};$$

$$f_E(x) = \tfrac{3}{4}\left[x(1+x^2)^{3/2} - \int_0^x (1+y^2)^{1/2}dy\right] - x^3$$
$$= \tfrac{3}{8}[x(1+x^2)^{1/2}(1+2x^2) - \log\{x+(1+x^2)^{1/2}\}] - x^3.$$

Let
$$f_1(x) = x(1+x^2)^{1/2}(1+2x^2) - \log\{x+(1+x^2)^{1/2}\}. \tag{12}$$

Then
$$f_E(x) = \tfrac{3}{8}f_1(x) - x^3. \tag{13}$$

$$P/A = f_P(x) = \int_0^x y^4(1+y^2)^{-1/2}dy. \tag{14}$$

Since
$$\frac{d}{dy}\{y^3(1+y^2)^{1/2}\} = y^4(1+y^2)^{-1/2} + 3y^2(1+y^2)^{1/2},$$
$$f_P(x) = x^3(1+x^2)^{1/2} - \tfrac{3}{8}f_1(x). \tag{15}$$

Expansions for x *Small and Large*

For x small, a series expansion for $f_E(x)$ and $f_P(x)$ may be obtained by noting that the log term is equal to $\int(1+x^2)^{-1/2}dx$. For x large, the log term may be neglected.

$x << 1$

$$f_E(x) = \tfrac{3}{10}x^5 - \tfrac{3}{56}x^7\cdots \tag{13'}$$

$$f_P(x) = \tfrac{1}{5}x^5 - \tfrac{1}{14}x^7\cdots \tag{15'}$$

These expressions give values correct to within 0·5 per cent. for $x \leqq 0{\cdot}4$.

$x >> 1$

$$f_E(x) = \tfrac{3}{4}x^2(x^2+1) - x^3\cdots \tag{13''}$$

$$f_P(x) = \tfrac{1}{4}x^2(x^2-1)\cdots \tag{15''}$$

These expressions give values correct to within 0·1 per cent. for $x > 10$.

It may be noticed that for $x << 1$, since $P/E = f_P(x)/f_E(x)$,

$$P = \tfrac{2}{3}E, \tag{16}$$

as for an ordinary gas ; while for $x >> 1$,

$$P = \tfrac{1}{3}E, \tag{16'}$$

the relativity effect then being predominant, and the same relation holding as for radiation.

Computation

The values of $f_P(x)$ and $f_E(x)$ have been computed by means of the sets of expressions (13) and (15). An accuracy within 0·5 per cent., which is probably amply sufficient for any applications that are likely to be made, could be easily attained with 4-figure logs, except in the range $x = 0{\cdot}4 - 0{\cdot}7$, for which 7-figure logs were used.

Numerical Values

The results for the energy and pressure are

$$E = Af_E(x), \tag{17}$$

$$P = Af_P(x), \tag{18}$$

where

$$x = \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3} = \alpha n^{1/3}, \tag{5}$$

$$A = \frac{8\pi(m_0 c)^5}{3m_0 h^3}. \tag{8}$$

The following values were used, obtained from Birge's table :—

$$h = 6{\cdot}547 \times 10^{-27},$$
$$c = 2{\cdot}998 \times 10^{10},$$
$$m_0 = 9{\cdot}035 \times 10^{-28}.$$

The main uncertainty is in connection with m_0. The spectroscopic value has been used, as the conditions when there are electrons at high concentration resemble those in atoms rather than those under which the deflection experiments are carried out. (The latter lead to $8{\cdot}994 \times 10^{-28}$ as the most probable value of m_0.) The above values lead to the following numerical relations between x and n:—

$$\left.\begin{aligned} x &= 1{\cdot}1901 \times 10^{-10} n^{1/3} \\ \log x &= \tfrac{1}{3}\log n - 10 + 0{\cdot}07558 \\ n &= 5{\cdot}932 \times 10^{29} x^3 \\ \log n &= 3\log x + 29{\cdot}7732 \\ \alpha &= 1{\cdot}1901 \times 10^{-10} \\ \log \alpha &= \bar{1}0{\cdot}07558 \end{aligned}\right\}. \tag{19}$$

[With $m_0 = 8{\cdot}994 \times 10^{-28}$ the results are

$$\left.\begin{aligned} x &= 1{\cdot}195 \times 10^{-10} n^{1/3} \\ \log x &= \tfrac{1}{3}\log n - 10 + 0{\cdot}0775 \\ n &= 5{\cdot}854 \times 10^{29} x^3 \\ \log n &= 3\log x + 29{\cdot}7675 \end{aligned}\right\}. \tag{19'}$$

In previous papers m_0 was taken as $9{\cdot}01 \times 10^{-28}$, giving $n = 5{\cdot}88 \times 10^{29} x^3$.]

The conditions approximate to ordinary degeneracy for $x << 1$, or $n << n'$, and to relativistic degeneracy for $x >> 1$, or $n >> n'$, where $n' = 5{\cdot}932 \times 10^{29}$.

Inserting numerical values,

$$\left.\begin{aligned} P = Af_P(x) &= 4{\cdot}818 \times 10^{23} f_P(x) \\ \log A &= 23{\cdot}6828 \end{aligned}\right\}. \tag{20}$$

To obtain the value of P (or E) for a given value of n, the simplest general procedure is to calculate the corresponding value of x, and then to find the value of $f_P(x)$ (or $f_E(x)$) by interpolation from the tables. It is, however, frequently convenient to use expressions in forms which show more clearly the relation to the limiting forms for low and high concentrations.

For

$$\begin{aligned} x << 1, \quad & f_P(x) = \tfrac{1}{5}x^5, \\ x >> 1, \quad & f_P(x) = \tfrac{1}{4}x^4, \end{aligned}$$

$$P = A\{f_P(x)/(\tfrac{1}{5}x^5)\}(\tfrac{1}{5}x^5) = A\{f_P(x)/(\tfrac{1}{4}x^4)\}(\tfrac{1}{4}x^4), \tag{21}$$

$$= (\tfrac{1}{5}A\alpha^5)\{f_P(x)/(\tfrac{1}{5}x^5)\}n^{5/3} = (\tfrac{1}{4}A\alpha^4)\{f_P(x)/(\tfrac{1}{4}x^4)\}n^{4/3}, \tag{22}$$

$$= \kappa_1' n^{5/3} \times \{f_P(x)/(\tfrac{1}{5}x^5)\} = \kappa_2' n^{4/3}\{f_P(x)/(\tfrac{1}{4}x^4)\}, \tag{23}$$

where

$$\left.\begin{aligned} \kappa_1' &= 2{\cdot}300 \times 10^{-27} \\ \log \kappa_1' &= \bar{2}\bar{7}{\cdot}3617 \\ \kappa_2' &= 2{\cdot}415 \times 10^{-17} \\ \log \kappa_2' &= \bar{1}\bar{7}{\cdot}3830 \end{aligned}\right\}. \tag{24}$$

Thus for n small and large, the results are

$$n << 5{\cdot}932 \times 10^{29}, \qquad P = 2{\cdot}300 \times 10^{-27} n^{5/3}, \tag{25}$$

$$n >> 5{\cdot}932 \times 10^{29}, \qquad P = 2{\cdot}415 \times 10^{-17} n^{4/3}. \tag{26}$$

The values of $f_P(x)/\frac{1}{5}x^5$ are tabulated over the range of x for which it is greater than 0·5 ; and the values of $f_P(x)/\frac{1}{4}x^4$ over the whole range, as this function has other applications.

Expressions for E similar to those for P may readily be obtained if required. For low and high concentrations the results are

$$n << 5{\cdot}932 \times 10^{29}, \qquad E = 3{\cdot}450 \times 10^{-27} n^{5/3}, \tag{27}$$

$$n >> 5{\cdot}932 \times 10^{29}, \qquad E = 7{\cdot}245 \times 10^{-17} n^{4/3}. \tag{28}$$

Pressure as a Function of Density

In many astrophysical applications it is convenient to express the pressure as a function of the density. Let ρ be the density and μ the mean molecular weight per electron. (This differs slightly from the usual use of μ, but is more convenient in the present connection.) Let c_A be the fractional

Table I

Functions for Calculating Kinetic Energy per Unit Volume, E, *and Pressure*, P, *in Terms of Electron Concentration*, n

$$n = 5{\cdot}932 \times 10^{29} x^3,$$
$$P = 4{\cdot}818 \times 10^{23} f_P(x) = 2{\cdot}300 \times 10^{-27} n^{5/3} f_P(x)/(\tfrac{1}{5}x^5) = 2{\cdot}415 \times 10^{-17} n^{4/3} f_P(x)/(\tfrac{1}{4}x^4),$$
$$E = 4{\cdot}818 \times 10^{23} f_E(x).$$

x	$\log n$	$f_E(x)$	$f_P(x)$	$f_P(x)/(\frac{1}{5}x^5)$	$f_P(x)/(\frac{1}{4}x^4)$
·1	26·7732	$2{\cdot}99 \times 10^{-6}$	$1{\cdot}99 \times 10^{-6}$	·996	·0796
·2	27·6762	$9{\cdot}53 \times 10^{-5}$	$6{\cdot}31 \times 10^{-5}$	·984	·158
·3	28·2045	$7{\cdot}17 \times 10^{-4}$	$4{\cdot}70 \times 10^{-4}$	·967	·232
·4	·5795	$2{\cdot}99 \times 10^{-3}$	$1{\cdot}94 \times 10^{-3}$	·947	·303
·5	·8702	$8{\cdot}99 \times 10^{-3}$	$5{\cdot}76 \times 10^{-3}$	·921	·369
·6	29·1078	$2{\cdot}20 \times 10^{-2}$	$1{\cdot}39 \times 10^{-2}$	·894	·429
·7	·3085	4·67	2·90	·863	·483
·8	·4825	8·92	5·47	·835	·534
·9	·6358	$1{\cdot}57 \times 10^{-1}$	9·44	·800	·575
1·0	·7732	2·60	$1{\cdot}54 \times 10^{-1}$	·770	·616
1·2	30·0108	6·18	3·53	·709	·682
1·4	·2115	$1{\cdot}27 \times 1$	7·04	·652	·733
1·6	·3855	2·37	$1{\cdot}26 \times 1$	·601	·774
1·8	·5391	4·06	2·12	·561	·807
2·0	·6762	6·55	3·33	·520	·832
3	31·2045	$3{\cdot}99 \times 10$	$1{\cdot}84 \times 10$	...	·908
4	·5795	$1{\cdot}39 \times 10^2$	6·07	...	·948
5	·8702	3·58	$1{\cdot}51 \times 10^2$	...	·964
6	32·1078	7·82	3·16	...	·975
7	·3085	$1{\cdot}49 \times 10^3$	5·89	...	·981
8	·4825	2·61	$1{\cdot}01 \times 10^3$	...	·985
9	·6358	4·25	1·62	...	·988
10	·7732	6·57	2·47	...	·990
20	33·6762	$1{\cdot}12 \times 10^5$	$3{\cdot}99 \times 10^4$	...	·997
30	34·2045	5·81	$2{\cdot}02 \times 10^5$	...	·999
40	·5795	$1{\cdot}86 \times 10^6$	6·40	...	1·000
50	·8702	4·56	$1{\cdot}56 \times 10^6$	...	1·000
60	35·1078	9·50	3·24	...	1·000
70	·3085	$1{\cdot}77 \times 10^7$	6·00	...	1·000
80	·4825	3·02	$1{\cdot}02 \times 10^7$	...	1·000
90	·6358	4·85	1·64	...	1·000
100	·7732	7·40	2·50	...	1·000

concentration (by mass) of atoms of atomic weight A, and atomic number Z, so that $\Sigma c_A = 1$. For completely ionised matter,

$$\mu = 1 \Big/ \left(\Sigma c_A \frac{Z}{A} \right). \tag{29}$$

(With this definition $\mu = 1$ for hydrogen, 2 for helium, 2·53 for lead, 1·43 for a mixture of 50 per cent. hydrogen and 50 per cent. lead.)

The density and electron concentration are then related by

$$\left.\begin{aligned} \rho &= \mu m_H n \\ &= 1{\cdot}6618 \times 10^{-24} \mu n \\ \log \rho &= \overline{24}{\cdot}22055 + \log \mu + \log n \\ n &= 6{\cdot}017 \times 10^{23} (\rho/\mu) \\ \log n &= 23{\cdot}77945 + \log \rho - \log \mu \end{aligned}\right\}. \tag{30}$$

As in dealing with pressure as a function of electron concentration, the simplest general procedure for finding the pressure for a given density is to calculate first the corresponding value of x, from which the value of $f_P(x)$ may be found from the tables, the pressure then being given by (20). The relation between ρ and x, using (19) and (30), is given by

$$\left.\begin{aligned} \rho &= 9{\cdot}856 \times 10^5 \mu x^3 \\ \log \rho &= 3 \log x + \log \mu + 5{\cdot}9937 \\ x &= 1{\cdot}005 \times 10^{-2} (\rho/\mu)^{1/3} \\ \log x &= \tfrac{1}{3} \log \rho - \tfrac{1}{3} \log \mu - 2 + 0{\cdot}0021 \end{aligned}\right\}. \tag{31}$$

The relation may be expressed as

$$\left.\begin{aligned} x &= c_1 \rho^{1/3} \\ \rho &= c_2 x^3 \end{aligned}\right\}, \tag{32}$$

c_1 and c_2 depending on μ as shown by (31), so that

$$\left.\begin{aligned} \log c_1 &= \bar{2}{\cdot}0021 - \tfrac{1}{3} \log \mu \\ \log c_2 &= 5{\cdot}9937 + \log \mu \end{aligned}\right\}. \tag{32'}$$

It may be noted that, for $x = 1$,

$$\rho = c_2 = \rho', \tag{33}$$

where ρ' corresponds to $x = 1$; that is, the conditions correspond to ordinary or to relativistic degeneracy according as $\rho << \rho'$ or $\rho >> \rho'$.

The equations corresponding to (23) are

$$\left.\begin{aligned} P &= \kappa_1 \rho^{5/3} \times \{ f_P(x) / (\tfrac{1}{5} x^5) \} \\ P &= \kappa_2 \rho^{4/3} \times \{ f_P(x) / (\tfrac{1}{4} x^4) \} \end{aligned}\right\}. \tag{34}$$

κ_1 and κ_2 depend on μ, and are related to κ_1' and κ_2' of (24) as follows:—

$$\left.\begin{aligned} \log \kappa_1 &= \log \kappa_1' - \tfrac{5}{3} \log m_H - \tfrac{5}{3} \log \mu \\ &= 12{\cdot}9941 - \tfrac{5}{3} \log \mu \\ \log \kappa_2 &= \log \kappa_2' - \tfrac{4}{3} \log m_H - \tfrac{4}{3} \log \mu \\ &= 15{\cdot}0889 - \tfrac{4}{3} \log \mu \end{aligned}\right\}. \tag{35}$$

The values of κ_1, κ_2, c_1 and c_2 (together with the logarithms, in brackets) are given in the following table for four values of μ:—

TABLE II

Values of Constants in the Formulæ

$P = \kappa_1 \rho^{5/3}$ for $\rho << \rho'$,
$P = \kappa_2 \rho^{4/3}$ for $\rho >> \rho'$,
and $x = c_1 \rho^{1/3}$.

(Logarithms are given in brackets)

μ	κ_1	$c_2 = \rho'$	κ_2	c_1
1·0	$9{\cdot}865 \times 10^{12}$ (12·9941)	$9{\cdot}856 \times 10^{5}$ (5·9937)	$1{\cdot}227 \times 10^{15}$ (15·0889)	$1{\cdot}005 \times 10^{-2}$ ($\bar{2}$·0021)
1·5	$5{\cdot}019 \times 10^{12}$ (12·7006)	$1{\cdot}479 \times 10^{6}$ (6·1698)	$7{\cdot}147 \times 10^{14}$ (14·8541)	$8{\cdot}778 \times 10^{-3}$ ($\bar{3}$·9434)
2·0	$3{\cdot}109 \times 10^{12}$ (12·4926)	$1{\cdot}971 \times 10^{6}$ (6·2947)	$4{\cdot}871 \times 10^{14}$ (14·6876)	$7{\cdot}977 \times 10^{-3}$ ($\bar{3}$·9018)
2·5	$2{\cdot}143 \times 10^{12}$ (12·3310)	$2{\cdot}463 \times 10^{6}$ (6·3916)	$3{\cdot}617 \times 10^{14}$ (14·5584)	$7{\cdot}404 \times 10^{-3}$ ($\bar{3}$·8695)

General Character of Results

With the aid of Tables I and II and the relevant equations, calculations of the pressure and electronic kinetic energy density corresponding to any value of the electron concentration, or of the material density (for $\mu = 1$ to $\mu = 2{\cdot}5$), may be made without much difficulty to a degree of accuracy which is probably amply sufficient for most purposes.

In order to show more immediately the general orders of the magnitudes involved, Table III has been drawn up. This shows the variation with n and ρ (for $\mu = 2$) of the pressure in the transition range. As will be obvious

TABLE III

Subsidiary Table, showing Values of Pressure (P) *for a Series of Values of the Electron Concentration* (n) *in the Transition Range ; with some Values obtained from the Simple Formulæ for Comparison*

log n	log ρ ($\mu = 2$)	a	P/a	$\kappa_1' n^{5/3}/a$	$\kappa_2' n^{4/3}/a$
27	0·5215 + 3	10^{18}	2·284	2·300	24·15
28	4	10^{20}	1·077		
29	5	10^{21}	4·490		
30	6	10^{23}	1·638	2·300	2·415
31	7	10^{24}	4·571		
32	8	10^{26}	1·089		
33	9	10^{27}	2·400	23·00	2·415

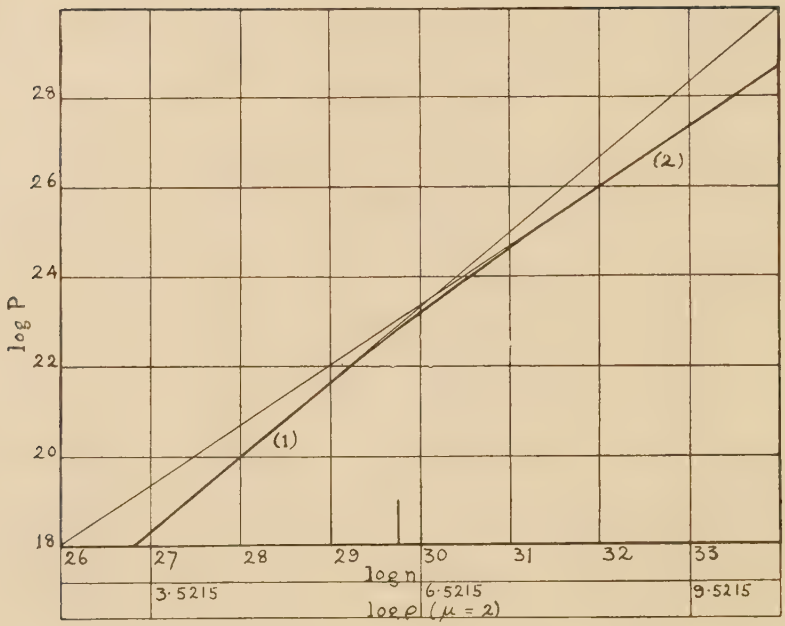

FIG. 1.—*Graph showing Relation between Pressure,* P, *and Electron Concentration,* n.

The curve gives log P as a function of log n.
The straight lines are log curves corresponding to

$$(1)\ P=\kappa_1' n^{5/3}=2{\cdot}300\times10^{-27}n^{5/3},$$
$$(2)\ P=\kappa_2' n^{4/3}=2{\cdot}415\times10^{-17}n^{4/3}.$$

The log n axis is also calibrated in terms of ρ for $\mu=2$.
The point marked is that corresponding to $n=5{\cdot}932\times10^{29}$, for which $x=1$.

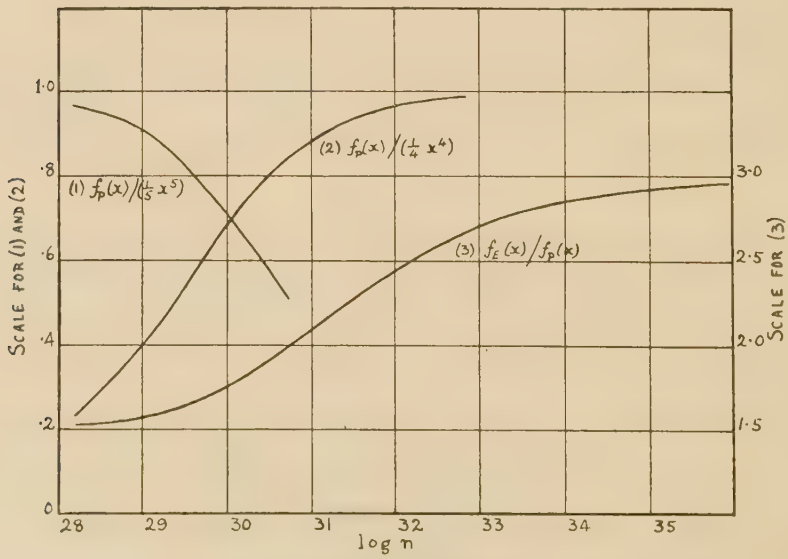

FIG. 2.—*Curves showing Relation between log* n *and*

(1) $f_P(x)/(\frac{1}{5}x^5)$, (2) $f_P(x)/(\frac{1}{4}x^4)$, (3) $f_E(x)/f_P(x)=E/P$.

For (1) and (2) let g_1, g_2 be the ordinates. Then

$$P=g_1\kappa_1' n^{5/3}=g_2\kappa_2' n^{4/3}.$$

from the comparative values given, below $n = 10^{27}$ the formula $P = \kappa_1' n^{5/3}$ (or $P = \kappa_1 \rho^{5/3}$) is adequate ; and above $n = 10^{33}$, the formula $P = \kappa_2' n^{4/3}$. The pressures given by the simple formulæ are always too great ; roughly, using the formula $P = \kappa_1' n^{5/3}$ *below* $n = 10^{30}$ increases the error to about 50 per cent. as n increases, while using $P = \kappa_2' n^{4/3}$ *above* $n = 10^{30}$ increases the error to about 50 per cent. as n decreases.

In fig. 1 a graph of log P against log n is shown ; the slope of the initial part being 5/3 and that of the final part, 4/3.

In fig. 2 the functions $f_P(x)/(\frac{1}{5}x^5)$ and $f_P(x)/(\frac{1}{4}x^4)$ are plotted. The curves show at once the extent of the deviation from the simple formulæ. A curve is also given showing the manner in which the pressure changes, owing to relativistic effect, from a value equal to two-thirds of the kinetic energy density at low concentrations to one-third at high concentrations.

Summary

In the investigation of the structure of white dwarfs and of the central regions of ordinary stars a knowledge of the pressure of matter at high densities is necessary. The pressure will be mainly due to the pressure of the " electron gas " subject to the Fermi-Dirac statistics. The minimum pressure is obtained by treating the gas as completely degenerate. Provided that the electron concentration, n, is not too large, the relativity change of mass with velocity may be neglected. Under these conditions, $P = \kappa_1' n^{5/3}$. When the relativity effect is predominant, $P = \kappa_2' n^{4/3}$. As the transition region corresponds to densities of astrophysical interest, the variation of P with n over the whole range from ordinary to relativistic degeneracy has been investigated.

A general expression for P (and also for E, the electronic kinetic energy density) is derived as an explicit function of x, where $x = \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} n^{1/3}$. The value of this function is tabulated for $x = 0{\cdot}1$ to $x = 100$ (Table I). This enables P to be calculated to within 1 per cent. for any value of n. For n small and large the results reduce to

$$P = 2{\cdot}300 \times 10^{-27} n^{5/3} \text{ for } n << n' \quad \text{and} \quad P = 2{\cdot}415 \times 10^{-17} n^{4/3} \text{ for } n >> n',$$

where $$n' = 5{\cdot}932 \times 10^{29}.$$

For a mean molecular weight (per electron) μ, the corresponding results in terms of density are $P = \kappa_1 \rho^{5/3}$ or $P = \kappa_2 \rho^{4/3}$, according as $\rho << \rho'$ or $\rho >> \rho'$. The values of κ_1, κ_2 and ρ' are calculated for $\mu = 1{\cdot}0$, $1{\cdot}5$, $2{\cdot}0$ and $2{\cdot}5$ (Table II).

The main features of the results are shown in two figures, one giving log P, the other the ratio of the pressure to the values calculated by the simple formulæ, and also E/P, as functions of log n.

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UPPER LIMITS FOR DENSITIES AND TEMPERATURES IN STARS.

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(Communicated by Sir A. S. Eddington)

An estimate of the maximum densities and temperatures possible in stars, if it could be made, would be of value for two main reasons. In the first place, although it could give no direct information as to the nature of the energy-generating processes which may occur in stars, the limiting conditions might restrict the kinds of processes which could reasonably be suggested. In the second place, although it can, by itself, give no general information about stellar structure, it might be of assistance in the physical interpretation of analytical results. Thus Milne * finds that, assuming that the material in stars behaves as a perfect gas, the analytical solutions of the equations, except under very limited conditions, give singularities at $r=0$, corresponding to his centrally condensed or collapsed models. The singularities, interpreted directly in terms of densities and temperatures, would be physically meaningless. Proceeding from the boundary to the centre of the star, a stage must be reached at which the perfect gas laws no longer hold, and a new equation of state must be used; the appropriate equation being approximately that corresponding to a degenerate electron gas. In Milne's hands the modification and development of the analysis for the perfect gas star on introducing the new equation leads to significant and far-reaching results for the general interpretation of stellar structure. Under certain conditions it may be necessary to introduce a further equation of state, corresponding to matter at the highest density of which it is capable, but this need not be considered here.

Milne's treatment is remarkable for its generality, and some of the conclusions to be developed here might be regarded simply as particular results which could be obtained, without much difficulty, from Milne's more general equations. There is, however, perhaps an advantage in separating out one particular aspect of the general problem—namely, that connected with the equation of state of highly ionised very dense matter—for in a very general treatment it is sometimes difficult to trace the origin from which certain particular and significant consequences follow. The present paper, then, is concerned mainly with the determination of values for the maximum density and temperature in stars of matter which is completely ionised, and the pressure of which can be treated as almost entirely due to the pressure of the degenerate electron gas. The method followed will be such that the densities and temperatures will be maxima for the matter in any state, provided only that the density does not decrease inwards in a star; it is

* E. A. Milne, *M.N.*, 91, 4, 1930, etc.

found, however, that maximum values can only be fixed by these particular considerations provided that the star has a mass below a critical value, which depends markedly on the mean molecular weight of the material of which it is composed, and also, of course, on the distribution of density. The occurrence of this critical mass is associated with the fact that the pressure, depending on the electron concentration, n, changes from variation as $n^{5/3}$ to variation as $n^{4/3}$ as the density increases, owing to the relativity change of mass with velocity. The variation of pressure with concentration has been determined over the whole range, and the results, given in a previous paper,* provide the data necessary for the present purpose.

The question of limiting densities in connection with white dwarf stars has already been discussed in a series of papers.† In the first two of these—the relativity effect being considered in the second—the case of a sphere of uniform density was considered. The results may be regarded as giving rough upper limits for the *mean* density. In the third paper the effect of non-uniform (polytropic) density distribution was discussed, some of the conclusions being similar to those reached by Chandrasekhar‡ at about the same time.

The present paper aims at the determination of maximum densities and also maximum temperatures. The method used is a development of that given by Eddington,§ who, however, considered only non-relativistic degeneracy. In a later paper Chandrasekhar|| has developed some of the consequences of the equations of state corresponding to ordinary and to complete relativistic degeneracy (without, however, considering the transition region), and certain of his results are closely connected with some of the conclusions arrived at here. Where there is overlapping, however, the method used here is rather different ; it can be made simpler and more direct owing to the less general aim of the investigation.

Determination of Maximum Density

In order that a star may be stable, the sum of the gas and radiation pressures at the centre must not exceed the downward gravitational pressure. Provided that the density does not decrease inwards, the maximum central gravitational pressure for a star of given mass is that for a sphere of uniform density, as shown by Eddington. (This case must be distinguished from that of a star of given mass *and radius*, for which uniform density gives the minimum central pressure.)

$$P_G + P_R \geqslant \frac{3}{8\pi}\frac{GM^2}{R^4}. \qquad (1)$$

The gravitational pressure may be expressed in terms of the mass and central

* E. C. Stoner, *M.N.*, **92**, 651, 1932.

† E. C. Stoner, *Phil. Mag.*, **7**, 63, 1929; **9**, 944, 1930. E. C. Stoner and F. Tyler, *ibid.*, **11**, 986, 1931.

‡ S. Chandrasekhar, *ibid.*, **11**, 592, 1931.

§ A. S. Eddington, *M.N.*, **91**, 444, 1931.

|| S. Chandrasekhar, *ibid.*, **91**, 456, 1931.

density. For the uniform sphere, $\rho = \rho_c = \bar{\rho}$. The gas pressure depends on the density, and also on the temperature. For completely ionised matter, the pressure will not be less than that of the degenerate electron gas at absolute zero. Let this pressure, a function of the density, be represented by $P(\rho)$. Then

$$\{P(\rho)\}_{\max} = \frac{3}{8\pi}\frac{GM^2}{R^4}$$

$$= \tfrac{1}{2}(\tfrac{4}{3}\pi)^{1/3}\, GM^{2/3}\, \rho^{4/3}. \tag{2}$$

Let
$$B_0 = \tfrac{1}{2}(\tfrac{4}{3}\pi)^{1/3}\, GM_s^{2/3}, \tag{3}$$

$$\{P(\rho)\}_{\max} = B_0 (M/M_s)^{2/3} \rho^{4/3}. \tag{2'}$$

Let μ be the mean molecular weight per electron of the material. $P(\rho)$ takes two limiting forms according as $(\rho/\rho') << 1$ or $>> 1$ where

$$\left.\begin{aligned} \rho' &= \frac{8\pi}{3}\left(\frac{m_0 c}{h}\right)^3 m_H \mu \\ &= 9{\cdot}856 \times 10^5 \mu \end{aligned}\right\}. \tag{4}$$

For $(\rho/\rho') << 1$,

$$\left.\begin{aligned} &P(\rho) = \kappa_1 \rho^{5/3}, \\ \text{where}\quad &\kappa_1 = \tfrac{1}{5}\left(\frac{3}{8\pi}\right)^{2/3} \frac{h^2}{m_0}\left(\frac{1}{\mu m_H}\right)^{5/3} \\ &= 9{\cdot}865 \times 10^{12}\left(\frac{1}{\mu}\right)^{5/3} \end{aligned}\right\}. \tag{5}$$

Under these conditions

$$\rho_{\max} = \left(\frac{B_0}{\kappa_1}\right)^3 \left(\frac{M}{M_s}\right)^2 \tag{6}$$

—the result obtained by Eddington.

For $(\rho/\rho') >> 1$,

$$\left.\begin{aligned} &P(\rho) = \kappa_2 \rho^{4/3}, \\ \text{where}\quad &\kappa_2 = \tfrac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3} hc\left(\frac{1}{\mu m_H}\right)^{4/3} \\ &= 1{\cdot}227 \times 10^{15}\left(\frac{1}{\mu}\right)^{4/3} \end{aligned}\right\}. \tag{7}$$

It follows that if the mass of a star is above a certain value the conditions considered do not suffice to determine a maximum density. The critical mass is given by

$$M_0/M_s = (\kappa_2/B_0)^{3/2}. \tag{8}$$

From (7) and (8) it will be seen that M_0 varies as $(1/\mu)^2$.

In general,

$$\left.\begin{aligned} P(\rho) = Af_P(x) &= \kappa_1 \rho^{5/3} f_P(x)/(\tfrac{1}{5}x^5) \\ &= \kappa_2 \rho^{4/3} f_P(x)/(\tfrac{1}{4}x^4) \end{aligned}\right\}, \tag{9}$$

where

$$\left.\begin{aligned} A &= \frac{8\pi(m_0 c)^5}{3m_0 h^3} \\ \kappa_1 &= \tfrac{1}{5} A \alpha^5 \left(\frac{1}{\mu m_H}\right)^{5/3} \\ \kappa_2 &= \tfrac{1}{4} A \alpha^4 \left(\frac{1}{\mu m_H}\right)^{4/3} \\ \alpha &= \frac{h}{m_0 c}\left(\frac{3}{8\pi}\right)^{1/3} \\ x &= \alpha\left(\frac{\rho}{\mu m_H}\right)^{1/3} \\ &= 1{\cdot}005 \times 10^{-2}\left(\frac{\rho}{\mu}\right)^{1/3} \\ f_P(x) &= x^3(1+x^2)^{1/2} - \tfrac{3}{8}[x(1+x^2)^{1/2}(1+2x^2) - \log\{x+(1+x^2)^{1/2}\}] \end{aligned}\right\}. \quad (10)$$

The values of $f_P(x)$ and also of $f_P(x)/(\frac{1}{4}x^4)$ have been tabulated for $x = 0{\cdot}1$ to $x = 100$.

Adopting the form most convenient for the present purpose,

$$P(\rho) = \kappa_2 \rho^{4/3} f_P(x)/(\tfrac{1}{4}x^4). \quad (11)$$

From (2′) and (11),

$$\left.\begin{aligned} \rho = \rho_{\max} \quad \text{for} \quad f_P(x)/(\tfrac{1}{4}x^4) &= \frac{B_0}{\kappa_2}\left(\frac{M}{M_s}\right)^{2/3} \\ &= \left(\frac{M_s}{M_0}\right)^{2/3}\left(\frac{M}{M_s}\right)^{2/3} \end{aligned}\right\}. \quad (12)$$

Thus for any value of M less than M_0 the value of $f_P(x)/(\frac{1}{4}x^4)$ corresponding to $\rho_{\max}$ may be found; then, from the tables previously given, the value of x. ρ is then found from

$$\left.\begin{aligned} \rho &= \mu m_H \left(\frac{m_0 c}{h}\right)^3 \left(\frac{8\pi}{3}\right) x^3 \\ &= 9{\cdot}856 \times 10^5 \mu x^3 \end{aligned}\right\}. \quad (13)$$

Relation to Previous Work

In the papers previously referred to,* the value of a maximum density was obtained by considering the change in the total electronic kinetic energy E_K and the gravitational energy E_G, for a star of given mass, with change of electron concentration, n. The equilibrium condition was put in the form

$$\frac{d}{dx}(E_G + E_K) = 0. \quad (14)$$

* E. C. Stoner, *Phil. Mag.*, **7**, 63, 1929; **9**, 944, 1930. E. C. Stoner and F. Tyler, *ibid.*, **11**, 986, 1931.

This led to the equation

$$\left.\begin{aligned} &\frac{20\pi}{3}\frac{m_0^4c^5}{h^3}x^4F(x)=B_0\left(\frac{M}{M_s}\right)^{2/3}\rho^{4/3}, \\ &\text{where} \\ &F(x)=\frac{d}{dx}\left[\frac{1}{x^3}\times\tfrac{1}{8}f_E(x)\right] \\ &\text{and} \\ &f_E(x)=\tfrac{3}{8}[x(1+x^2)^{1/2}(1+2x^2)-\log\{x+(1+x^2)^{1/2}\}]-x^3 \end{aligned}\right\}. \tag{15}$$

The value of $F(x)$ was tabulated for a series of values of x. It may be shown that

$$F(x)=\tfrac{1}{4}\times f_P(x)/(\tfrac{1}{4}x^4). \tag{16}$$

Thus the condition used for determining the maximum density was

$$\tfrac{5}{2}Af_P(x)=B_0\left(\frac{M}{M_s}\right)^{2/3}\rho^{4/3}, \tag{17}$$

to be compared with the present condition,

$$Af_P(x)=B_0\left(\frac{M}{M_s}\right)^{2/3}\rho^{4/3}. \tag{18}$$

Now for a star of uniform density,

$$\overline{P}=\frac{\int P dv}{\int dv}=\frac{\frac{1}{3}\Omega}{\frac{4}{3}\pi R^3}=\frac{3}{20\pi}\frac{GM^2}{R^4}=\tfrac{2}{5}P_c. \tag{19}$$

In effect, an estimate of the maximum density was made by using the condition that the pressure of the electron gas (uniform, since uniform density was assumed) should not exceed the *average* gravitational pressure, and, as previously stated, the method was suitable for gaining an approximate estimate of the maximum average density. For $\rho << \rho'$, $Af_P(x)=\kappa_1\rho^{5/3}$, and the density found, as is obvious from (17) and (18), is $(2/5)^3$ smaller than by the present method, while the critical mass is $(5/2)^{3/2}$ larger, for the same value of μ. While the method used served its purpose in giving a first clear indication of character of the results following from a consideration of the properties of a degenerate electron gas, it does not give strict upper limits for the density and it is not well adapted for direct further development and application to models approximating more closely to actual stars.

Determination of Maximum Temperature

Provided that the mass of a star is below the critical value, a maximum temperature may also be fixed by the conditions of equation (1), which may be written

$$P(\rho)+\tfrac{1}{3}aT^4 \not> B_0(M/M_s)^{2/3}\rho^{4/3}. \tag{20}$$

Since $P(\rho)$ is the minimum pressure of ionized matter at the density ρ,

$$\tfrac{1}{3}aT^4 \not> B_0(M/M_s)^{2/3}\rho^{4/3}-P(\rho). \tag{21}$$

The maximum value of T occurs when the right-hand side is a maximum. For

$$\rho << \rho', \qquad P(\rho) = \kappa_1 \rho^{5/3},$$

$$T = T_{\max} \quad \text{for} \quad \frac{d}{d\rho}[B_0(M/M_s)^{2/3}\rho^{4/3} - \kappa_1\rho^{5/3}] = 0,$$

$$\rho = \left(\frac{4}{5}\frac{B_0}{\kappa_1}\right)^3 \left(\frac{M}{M_s}\right)^2. \tag{22}$$

Substituting in (21),

$$T_{\max} = \left(\frac{3}{5a}\right)^{1/4} \frac{1}{\kappa_1}\{B_0(M/M_s)^{2/3}\}^{5/4}. \tag{23}$$

This is Eddington's result.

In deriving the general result it is convenient to use x, defined by (10) or (13), as the variable in place of ρ.

$$\rho = \frac{\mu m_H}{a^3}x^3. \tag{24}$$

$$P(\rho) = Af_P(x) = 4\kappa_2\left\{\frac{(\mu m_H)^{1/3}}{a}\right\}^4 f_P(x). \tag{25}$$

Substituting in (21), and making use of (8),

$$T = T_{\max} \quad \text{for} \quad \frac{d}{dx}[B_0(M/M_s)^{2/3}x^4 - 4\kappa_2 f_P(x)] = 0,$$

$$\frac{\frac{d}{dx}f_P(x)}{x^3} = \frac{B_0(M/M_s)^{2/3}}{\kappa_2}$$

$$= \frac{(M/M_s)^{2/3}}{(M_0/M_s)^{2/3}} = \left(\frac{M}{M_0}\right)^{2/3}. \tag{26}$$

Now

$$f_P(x) = \int_0^x y^4(1+y^2)^{-1/2}dy, \tag{27}$$

$$\frac{\frac{d}{dx}f_P(x)}{x^3} = \frac{x}{(1+x^2)^{1/2}}, \tag{28}$$

giving

$$T = T_{\max} \quad \text{for} \quad x^2 = \frac{1}{(M_0/M)^{4/3} - 1} \tag{29}$$

$$= \frac{\{B_0(M/M_s)^{2/3}\}^2}{\kappa_2{}^2 - \{B_0(M/M_s)^{2/3}\}^2}.$$

The value of x corresponding to $T_{\max}$ may thus be determined for any value of M less than M_0. Using this value of x in (21),

$$T_{\max} = \left(\frac{3}{a}\right)^{1/4}\left(\frac{\mu m_H}{a^3}x^3\right)^{1/3}\left[B_0(M/M_s)^{2/3} - \kappa_2\frac{f_P(x)}{\frac{1}{4}x^4}\right]^{1/4}. \tag{30}$$

The corresponding value of ρ is

$$\rho = \frac{\mu m_H}{a^3}x^3. \tag{31}$$

Numerical Results

Calculations have been made for four values of the mean molecular weight per electron, $\mu = 1{\cdot}0$, $1{\cdot}5$, $2{\cdot}0$ and $2{\cdot}5$. The relevant degenerate electron gas constants, κ_1 and κ_2, have already been obtained and tabulated.* The chief remaining constants are B_0 and a, Stefan's constant. The values given by Eddington have been used.

$$\left.\begin{aligned} B_0 &= \tfrac{1}{2}(\tfrac{4}{3}\pi)^{1/3} G M_s^{2/3} \\ &= 8{\cdot}478 \times 10^{14} \\ \log B_0 &= 14{\cdot}9283 \end{aligned}\right\}. \tag{32}$$

$$\left.\begin{aligned} a &= 7{\cdot}64 \times 10^{-15} \\ \log a &= \bar{15}{\cdot}8832 \end{aligned}\right\}. \tag{33}$$

Critical Mass.—This is found directly from (8),

$$\log (M_0/M_s) = \tfrac{3}{2}\{\log \kappa_2 - \log B_0\}. \tag{34}$$

Maximum Density.—From (12),

$$\rho = \rho_{\max} \quad \text{for} \quad f_P(x)/(\tfrac{1}{4}x^4) = (M/M_s)^{2/3}(M_s/M_0)^{2/3}.$$

The value of $f_P(x)/(\frac{1}{4}x^4)$ is calculated for a series of values of (M/M_s) up to (M_0/M_s). The corresponding value of x is then found by interpolation from the tabulated values. With this value of x,

$$\rho_{\max} = 9{\cdot}856 \times 10^5 \mu x^3,$$

$$\log \rho_{\max} = 3 \log x + 5{\cdot}9937 + \log \mu. \tag{35}$$

Maximum Temperature.—From (29),

$$T = T_{\max} \quad \text{for} \quad x^2 = \frac{(M/M_0)^{4/3}}{1 - (M/M_0)^{4/3}},$$

$$\log x = \tfrac{2}{3} \log (M/M_0) - \tfrac{1}{2} \log [1 - (M/M_0)^{4/3}]. \tag{36}$$

The value of $f_P(x)/(\frac{1}{4}x^4)$ for this value of x is found by interpolation from the tables. Then from (20),

$$T_{\max} = \left(\frac{3}{a}\right)^{1/4} \frac{(\mu m_H)^{1/3}}{a} x \left[B(M/M_s)^{2/3} - \kappa_2 \frac{f_P(x)}{\frac{1}{4}x_4} \right]^{1/4},$$

$$\log T_{\max} = 5{\cdot}6464 + \log x + \tfrac{1}{3} \log \mu + \tfrac{1}{4} \log [B(M/M_s)^{2/3} - \kappa_2 f_P(x)/(\tfrac{1}{4}x^4)]. \tag{37}$$

For the corresponding value of ρ, from (31),

$$\log \rho = 5{\cdot}9937 + 3 \log x + \log \mu. \tag{38}$$

The results of the computations are shown in the following tables and curves, which show the logarithms of $\rho_{\max}$ and of $T_{\max}$ (and of ρ for $T_{\max}$) as functions of M/M_s. It may be noted that the curves for $\log \rho$ are of the same form. If the M/M_s axis is recalibrated so that $(M_0/M_s)_{\mu=1}$ $(= 1{\cdot}742)$ is given the value 1, the $\mu = 1$ curve becomes a "reduced curve" giving $\log (\rho/\mu)$ as a function of $(M/M_0)_\mu$.

* E. C. Stoner, *M.N.*, **92**, 651, 1932.

TABLE I

Logarithms of ρ_{max}, T_{max} *and* ρ *for* T_{max} *for Values of* (M/M_s) *up to* (M_0/M_s) *for Different Values of the Mean Molecular Weight per Electron,* μ.

M/M_s	ρ_{max}	T_{max}	ρ for T_{max}
	$\mu=1$ $M_0/M_s=1\cdot742$		
·1	3·8163	8·2151	3·5362
·2	4·4466	·4659	4·1512
·3	4·8258	·6233	4·5318
·4	5·1165	·7403	4·8150
·5	5·3541	·8222	5·0463
·6	5·5602	·9021	5·2485
·7	5·7447	·9685	5·4318
·8	5·9160	9·0300	5·6031
·9	6·0939	·0959	5·7687
1·0	6·2634	·1386	5·9346
1·1	6·4413	·1929	6·1032
1·2	6·6381	·2461	6·2720
1·3	6·8436	·3054	6·4929
1·4	7·1235	·3640	6·7011
1·5	7·4031	·4424	6·9799
1·6	7·7670	·5388	7·3764
1·7	8·6865	·7714	8·4481

ρ_{max}	T_{max}	ρ for T_{max}	M/M_s
	$\mu=1\cdot5$ $M_0/M_s=\cdot7740$		
4·7331	8·5124	4·4364	·1
5·4144	8·7807	5·1114	·2
6·8758	8·9508	5·5632	·3
6·2691	9·0875	5·9460	·4
6·6702	9·2052	6·3234	·5
7·1550	9·3346	6·7611	·6
7·8387	9·5166	7·4370	·7
8·6673	9·6610	8·2203	·75
	$\mu=2\cdot0$ $M_0/M_s=\cdot4356$		
5·4175	8·7347	5·1151	·1
6·2209	9·0290	5·9032	·2
6·9391	9·2448	6·5803	·3
8·0680	9·5389	7·6768	·4
	$\mu=2\cdot5$ $M_0/M_s=\cdot2787$		
6·0028	8·9182	5·6929	·1
7·0753	9·2953	6·7732	·2
8·4079	9·5055	7·6003	·25

M_0 = critical mass. To obtain the values of (M/M_s) for different polytropic density distributions for the same values of ρ_{max} and T_{max}, the tabulated values of (M/M_s) must be multiplied by 2·192 for $n=3/2(\gamma=5/3)$ and by 3·291 for $n=3(\gamma=4/3)$.

Density Distribution

The consideration of a model of uniform density enables estimates to be made of extreme upper limits (under the conditions indicated) for densities and temperatures in stars; but the state of uniform density is one which can hardly be approached in actual stars except, possibly, in the central regions when the densities are much greater than those discussed here. To obtain estimates which, although less certain as upper limits, are probably more appropriate to actual stars, it is reasonable to consider density distributions which are compatible with the equations of state corresponding to a degenerate electron gas, namely $P=\kappa_1\rho^{5/3}$ or $\kappa\rho_2{}^{4/3}$ according as $\rho >> $ or $<< \rho'$ (ρ' being of the order 10^6). Compatible density distributions are those corresponding to the Emden polytropes, $n=3/2$ and $n=3$. If degenerate electron-gas pressure only had to be considered, the polytrope of index 3/2

would be appropriate for $\rho_c < O(10^6)$ and a composite polytrope for $\rho_c > O(10^6)$ with index 3 for the inner region. The problem, however, is complicated by

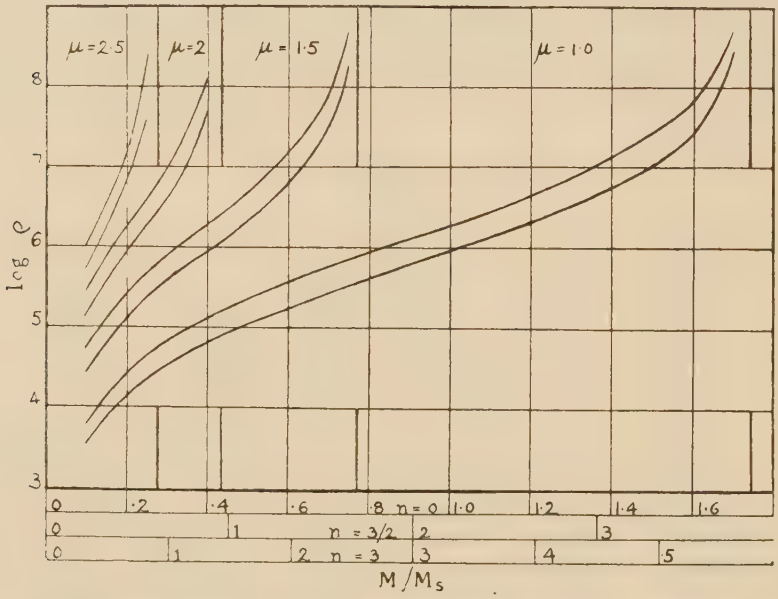

FIG. 1.—*Curves showing relation between stellar mass and* (1) $\log \rho_{\max}$ (*upper curves*) *and* (2) $\log \rho$ *for* $T_{\max}$ (*lower curves*) *for four values of* μ. *The* M/M_8 *axis is calibrated for three different density distributions, corresponding to the polytropic indices* $n=0(\gamma=\infty$, *uniform density*), $n=3/2(\gamma=5/3)$ *and* $n=3(\gamma=4/3)$.

the radiation pressure; but even considering this, it seems reasonable to regard the $n=3/2$ and $n=3$ polytropes as bounding the density distribution

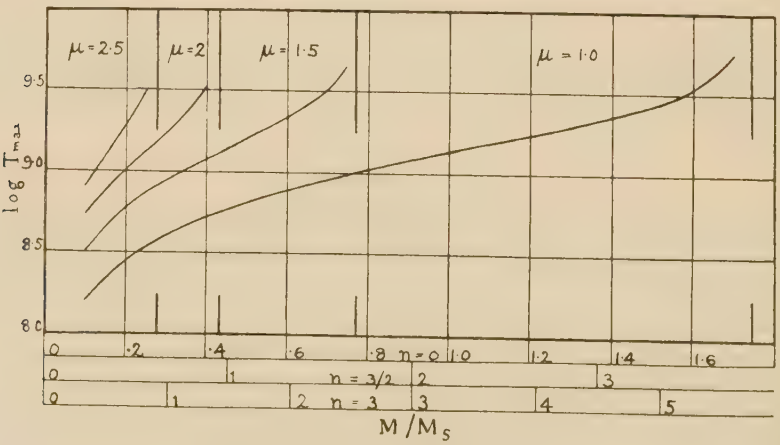

FIG. 2.—*Curves showing relation between stellar mass and maximum temperature.*

appropriate to completely ionised matter. The effect of these two distributions on the upper limits for the central density will then be considered, as it is believed that this procedure enables more definite general conclusions

—at least at the present stage—to be drawn than the consideration of some arbitrarily composite polytrope.

The equation (2) may now be written

$$\{P(\rho)\}_{\max} = B_n(M/M_s)^{2/3}\rho^{4/3}, \tag{39}$$

where B_n is the coefficient for the polytrope of index n, replacing B_0 for the uniform density sphere, and ρ is the central density.

To determine B_n a value is required for the gravitational pressure P_0 at the centre of the polytrope. With the symbols employed by Eddington * in his treatment of the polytropic equations—using R' and M' as the tabulated reduced values corresponding to the radius and the mass—

$$P_0 = \left\{\frac{\phi_0}{n+1}\right\}^{n+1}\frac{1}{\kappa^n},$$

$$\phi_0 = G\frac{M}{M'}\frac{R'}{R},$$

$$\frac{\{(n+1)\kappa\}^n}{4\pi G} = \left(\frac{GM}{M'}\right)^{n-1}\left(\frac{R'}{R}\right)^{n-3},$$

giving

$$P_0 = \frac{1}{4\pi(n+1)}G\left(\frac{M}{M'}\right)^2\left(\frac{R'}{R}\right)^4. \tag{40}$$

Let ρ_0 be the central density, $\bar{\rho}$ the mean density.

$$P_0 = \tfrac{1}{2}\left(\frac{4\pi}{3}\right)^{1/3}GM_s^{2/3}(M/M_s)^{2/3}\rho_0^{4/3}\left[\frac{2}{3(n+1)}\frac{R'^4}{M'^2}\left(\frac{\bar{\rho}}{\rho_0}\right)^{4/3}\right]$$

$$= B_0(M/M_s)^{2/3}\rho_0^{4/3}\left[\frac{2}{3(n+1)}\frac{R'^4}{M'^2}\left(\frac{\bar{\rho}}{\rho_0}\right)^{4/3}\right],$$

$$B_n = \left[\frac{2}{3(n+1)}\frac{R'^4}{M'^2}\left(\frac{\bar{\rho}}{\rho_0}\right)^{4/3}\right]B_0, \tag{41}$$

$n = 3/2 (\gamma = 5/3)$

$$M' = 2{\cdot}717, \qquad \rho_0/\bar{\rho} = 6{\cdot}0, \qquad R' = 3{\cdot}657$$

$$B_{3/2} = {\cdot}5926B_0 = B_0/1{\cdot}687, \tag{42}$$

$n = 3 (\gamma = 4/3)$

$$M' = 2{\cdot}015, \qquad \rho_0/\bar{\rho} = 54{\cdot}36, \qquad R' = 6{\cdot}9011$$

$$B_3 = {\cdot}4520B_0 = B_0/2{\cdot}212. \tag{43}$$

Critical Mass.—If the density increases inwards, the critical mass (below which the present conditions are sufficient to fix an upper limit to the density) will be greater than for uniform density. Modifying equation (8),

$$(M_0/M_s)_n = (\kappa_2/B_n)^{3/2}$$

$$= (B_0/B_n)^{3/2}(M_0/M_s)_0. \tag{44}$$

* A. S. Eddington, *The Internal Constitution of the Stars*, chap. iv, particularly pp. 80 and 83.

The values for the critical mass are shown in the following table:—

TABLE II

Values of M_0/M_s (M_0 = *critical mass*) *for Different Values of* μ (*mean molecular weight per electron*) *and Density Distribution.* (n = *polytropic index.*)

μ \ n	0 ($\gamma=\infty$)	3/2 ($\gamma=5/3$)	3 ($\gamma=4/3$)
1·0	1·742	3·817	5·731
1·5	·7740	1·347	2·546
2·0	·4356	·9545	1·433
2·5	·2787	·6106	·9168

Maximum Density.—For $\rho_c << \rho'$ the maximum density is decreased (relative to the value for uniform density) as shown by (6) in the ratio $(B_n/B_0)^3$ for a given mass. For $n=3/2$, the maximum density is 4·80 times as small; for $n=3$, 10·8 times as small.

In general, the values for the maximum density may readily be derived from those for a sphere of uniform density. Adapting equation (12),

$$\rho=\rho_{\max} \quad \text{for} \quad \frac{f_P(x)}{\frac{1}{4}x^4}=\left(\frac{M}{M_0}\right)_n^{2/3}. \tag{45}$$

Thus for a distribution corresponding to the polytropic index n, the maximum density will be the same for a mass $(M)_n$ as for a mass $(M)_0$ for a sphere of uniform density (index $n=0$), where

$$(M)_n=(M)_0\times(M_0)_n/(M_0)_0=(M)_0\times(B_0/B_n)^{3/2}. \tag{46}$$

To obtain numerical values, the values of (M/M_s) in Table I must be multiplied by $(M_0)_n/(M_0)_0$ or $(B_0/B_n)^{3/2}$ to give the values of $(M/M_s)_n$ for the polytropic index n distribution for the same maximum density.

In the graphs giving the relation between $\log\rho_{\max}$ and (M/M_s) the unit of the abscissa scale has simply to be increased in the appropriate ratio to convert the curves into those for the new distribution.

TABLE III

Multiplication Factors for (M/M_s) *in Table I, and Units for Graphs of Figures* 1 *and* 2 *for Different Density Distributions*

n	0	3/2	3
$(B_0/B_n)^{3/2}=(M_0)_n/(M_0)_0$	1	2·192	3·291

Maximum Temperature.—A precisely similar method may be employed for deriving the maximum temperature for different polytropic distributions

from that for uniform density as in dealing with the maximum density. Adapting equation (36),

$$T = T_{\max} \quad \text{for} \quad x^2 = \frac{(M/M_0)_n^{4/3}}{1 - (M/M_0)_n^{4/3}}. \tag{47}$$

The value of x (and so of the density) for which $T = T_{\max}$ will be the same for a mass $(M)_n$ as for a mass $(M)_0$ for a sphere of uniform density, where $(M)_n$ and $(M)_0$ are related by (46),

$$(M)_n = (M)_0 \times (M_0)_n/(M_0)_0.$$

Moreover, the maximum temperature will be the same for this mass, since in the expression (37) x is the same and, using (44) and (46),

$$B(M/M_s)^{2/3} = B_0(M/M_s)_0^{2/3} = B_n(M/M_s)_n^{2/3}.$$

Thus Table I may also be used to determine the maximum temperatures (and the corresponding densities) for stars with density distributions corresponding to $n = 3/2$ and $n = 3$, by multiplying the values of M/M_s by 2·192 and 3·291 respectively.

In figs. 1 and 2 the (M/M_s) axis has been calibrated for the three values of the polytropic index n which have been considered.

Mean Molecular Weight and Critical Mass

The main purpose of this inquiry has been to find how far upper limits to the density and temperature of matter in stars are imposed by the properties of a degenerate electron gas, on which the pressure of very dense matter mainly depends; and to determine these maximum densities and temperatures quantitatively. The answers to the questions involved are provided by the above results. The computations have been made for a number of values of the mean molecular weight, μ, and for a series of different density distributions in the star, corresponding to different values of a polytropic index, n. The aim has been to give the results in such a form that their consequences for more particularised conditions can be readily seen. Anything approaching a full discussion of what these conditions might be would soon lead to an extensive incursion into the general theory of stellar structure, which would here be out of place. It is, however, perhaps desirable to add a few remarks on the questions connected with the mean molecular weight and the critical mass.

Mean Molecular Weight.—The range of stellar masses to which the present results have direct application depends markedly on the value of the mean molecular weight per electron (see Table II). With the definition of μ here used,

$$\mu = 1/\Sigma c_A \frac{Z}{A}, \tag{48}$$

where c_A is the fractional concentration, by mass, of elements of atomic weight A and atomic number Z. When Z is small, the values of μ differ from those following from the more usual definition (mean molecular weight

per particle), being greater in the ratio $(Z+1)/Z$. The following table gives the values of μ for a number of elements and some mixtures:—

	H	*He*	*Pb*	*U*	50 per cent. *H* +50 per cent. *Pb*	30 per cent. *H* +70 per cent. *Pb*
μ	1·00	2·00	2·53	2·59	1·43	1·73

The question arises as to what is the most reasonable value to assume for μ in stars in the lower ranges of the stellar mass scale. (Even for a pure hydrogen star, the critical mass, with the most favourable assumptions, is well below the mass of the more massive stars—see Table II.) It would be idle to make dogmatic assertions on this point. If a star is mainly composed of heavy elements (at least in the inner parts) a value $\mu = 2{\cdot}5$ may be assumed. If Jeans's hyperuranium atoms were present, the value might be slightly increased, though even adopting that hypothesis, in the stars of smaller mass (and smaller luminosity) such atoms would presumably have largely been transformed into ordinary atoms. The mean molecular weight is appreciably decreased only if there is a considerable amount of hydrogen present. The problem of the hydrogen content of stars has recently been considered thoroughly by Strömgren,* who concludes that many of the difficulties in connexion with the astronomical opacity coefficient disappear if there is a fairly large hydrogen content, differing for different stars, but amounting to about 30 per cent. for three stars considered in detail (Capella, the Sun, Sirius). Such a hydrogen content gives 1·73 for μ, but this of course would not hold generally, nor can it be concluded at present that high hydrogen content is the only means of overcoming the opacity difficulties. There is, however, some justification for assuming that μ, in general, will not be as great as 2·5, and that it will certainly be greater than 1·5. The value $\mu = 2$ seems a reasonable average value to assume—to be decreased if further evidence suggests that the hydrogen content in particular stars is large, and increased if it is small.

The Critical Mass.—It has not been a primary aim to determine a mass which is critical in a mass sequence of stellar models. The critical mass $(M_0)_n$ which appears here is simply a mass such that for $M < (M_0)_n$, and a density distribution corresponding to an Emden polytrope of index n, there is a finite upper limit to the central density and temperature, conditioned by the pressure of the degenerate electron gas. For a star in which this is the controlling pressure the curve relating $\rho_{\max}$ and M/M_s will change from that for $n = 3/2$ to that for $n = 3$ as ρ changes from $\rho << \rho'$ to $\rho >> \rho'$, where $\rho' = 9{\cdot}856 \times 10^5$—and so as M/M_s increases. The value of $(M_0)_3$ therefore fixes the upper limit to stellar masses for which the present considerations are sufficient to fix upper limits for ρ and T. Roughly speaking, the curve relating $\rho_{\max}$ (or $T_{\max}$) with M/M_s may be said to have significance at its lower end for $n = 3/2$ and at its upper end for $n = 3$; so that $(M_0)_{3/2}$ has only an artificial (analytical) significance. The curve for $n = 0$, and the value of $(M_0)_0$ have significance only in giving extreme upper limits, and in defining the range of masses in which these can be fixed.

* B. Strömgren, *Zeits. für Astrophys.*, 4, 118, 1932.

Chandrasekhar,* in his work on highly collapsed configurations, has obtained and discussed the significance of two critical masses, namely,

$$M_{3/2} = \cdot 61 M_s \beta^{-3/2} \tag{49}$$

and

$$M_3 = \cdot 92 M_s \beta^{-3/2}, \tag{50}$$

where β is the ratio of the gas pressure to the total pressure, and is expressible in terms of the opacity, luminosity and mass. The value $\mu = 2 \cdot 5$ has been used, and the coefficients of $M_s\beta^{-3/2}$ agree with the values obtained here for $(M_0/M_s)_{3/2}$ and $(M_0/M_s)_3$. The factor $\beta^{-3/2}$ "takes care of" the radiation-pressure. It should be noted that for determining *maximum* densities for *given* masses it is legitimate to put $\beta = 1$, as is done here. In determining maximum temperatures, the explicit expression for the radiation-pressure ($\frac{1}{3}aT^4$) was used, and the temperatures obtained are also maximum values for given masses. The effect of inclusion of the β factor is simply to increase the range of mass to which the limiting conditions have application. (It is immediately obvious that the addition of radiation pressure to the electron-gas pressure will increase the mass which can be supported.)

The limit M_3 given by Chandrasekhar, then, agrees with that obtained here, and is arrived at by essentially the same argument. He interprets $M_{3/2}$ as the limiting mass for the electron gas to be in an ordinary (non-relativistic) degenerate state, and it is not immediately obvious why there is numerical agreement between the factor $\cdot 61$ and the value obtained here for $(M_0/M_s)_{3/2}$. The agreement comes about in the following way. Chandrasekhar assumes that the limiting $n = 3/2$ polytropic distribution will occur when the central density is ρ'', where

$$\kappa_1(\rho'')^{5/3} = \kappa_2(\rho'')^{4/3}, \tag{51}$$

and that these expressions give the pressure corresponding to ρ''. Neglecting the β factor, for the limiting mass it follows from (39) that

$$\kappa_2(\rho'')^{4/3} = B_{3/2}(M/M_s)^{2/3}(\rho'')^{4/3},$$

giving

$$(M/M_s) = (\kappa_2/B_{3/2})^{3/2}$$

in agreement with the values of $(M_0/M_s)_{3/2}$ obtained here (equation 44).

It is clear that Chandrasekhar's criterion is not a very strict one. In the first place ρ'' differs from ρ' (it may be shown that $\rho'' = (5/4)^3\rho'$), and in the second the actual pressure for $\rho = \rho''$ is some 50 per cent. less than that given by $P = \kappa_2(\rho'')^{4/3}$. It is, in fact, not possible to fix a *single* definite limit below which $P = \kappa_1\rho^{5/3}$ holds, and above which $P = \kappa_2\rho^{4/3}$. None the less, if a very rough approximate value is required below which ordinary degeneracy is predominant $(M_0/M_s)_{3/2}$ may be used in the expression (49).

The important critical mass, then, is $(M_0)_3$. For values of M greater than $(M_0)_3\beta^{-3/2}$ there is nothing in the present conditions to prevent indefinitely high central densities and temperatures being attained. Now the *lower* limit for this critical mass is $0 \cdot 92 M_s$, and it may be considerably higher if there is any considerable amount of hydrogen present (Table II). For

* S. Chandrasekhar, *M.N.*, **91**, 456, 1931.

stars of mass below this (which will include white dwarfs, and a considerable proportion of ordinary dwarfs), provided the mass is not too close to the limit, there are rough upper limits to the central density of 5×10^8, and to the central temperature of 5×10^9. (See Table I for more precise values.) These, then, define the conditions under which energy-generating processes in these stars occur. The fact that there are even in some 'ordinary' stars definite limits to temperature and density is perhaps important, even though the restrictions are not very stringent except for stars of very low mass ; and it is of interest that these restrictions do not hold for the luminous massive stars.

Summary.

Upper limits for the central densities and temperatures of stars are obtained by using the inequality

$$P(\rho) + \tfrac{1}{3}aT^4 \not> B(M/M_s)^{2/3}\rho^{4/3},$$

where $P(\rho)$ is the pressure due to a completely degenerate electron gas. The relation to previous work is indicated. The values of $P(\rho)$ (which has the limiting forms $\kappa_1\rho^{5/3}$ and $\kappa_2\rho^{4/3}$ for $\rho << \rho'$ and $\rho >> \rho'$, where $\rho' = 9{\cdot}856 \times 10^5$) are obtained from a previous paper. The value of B is found for a sphere of uniform density (B_0) and for density distributions corresponding to the Emden polytropes $n = 3/2$ ($\gamma = 5/3$) and $n = 3$ ($\gamma = 4/3$). For masses below M_0, where $(M_0/M_s)_n = (\kappa_2/B_n)^{3/2}$, upper limits for ρ and T can be determined. These have primarily been computed for a sphere of uniform density—this giving extreme upper limits—for mean molecular weights per electron (μ) of 1·0, 1·5, 2·0 and 2·5. (Table I and figs. 1 and 2.)

It is then shown that the results for density distributions corresponding to $n = 3/2$ and $n = 3$ can be readily derived from those given ; for the same maximum densities and temperatures, the values of M/M_s (including M_0/M_s) must be multiplied by 2·192 and 3·291 respectively. As M increases through $(M_0)_{3/2}$ to $(M_0)_3$ the relations between mass and $\rho_{\max}$ (or $T_{\max}$) change gradually from those for $n = 3/2$ to those for $n = 3$. The values of M_0/M_s for different values of n and μ are shown in Table II. The method enables upper limits for T and ρ to be fixed for stars of mass less than $(M_0)_3$. The value of μ is briefly considered, and the significance of the critical masses is discussed.

The general conclusion is that for stars of mass not too close to a value whose lower limit is $0{\cdot}92M_s$ there is an upper limit to the central density of the order 5×10^8, and to the central temperature of the order 5×10^9, more precise values for a range of masses being given in the tables.

I am greatly indebted to Sir Arthur Eddington for suggesting the desirability of this more detailed investigation of the question of "upper limits," and for his interest in the progress of the work.

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1932 *April.*

ON THE TEMPERATURES OF WOLF-RAYET STARS AND NOVÆ.

C. S. Beals.

Introduction

The temperatures of the hotter stars (O and early B types) are not at present known with any degree of certainty. The theory of thermal ionization when applied to absorption-line intensities in early B stars yields values ranging from 15,000° for class B5 to 20,000° for class Bo.* Determinations of colour temperature based on colour indices † and on measures of the distribution of energy in the continuous spectrum ‡ lead to lower values. If the results of various observers be combined we arrive at an approximate temperature of 13,000° for the group Bo to B5, considerably lower than the ionization temperatures. Not only are the colour temperatures lower but they do not indicate the progression shown by the ionization temperatures in passing along the sequence from Bo to B5. Hertzsprung finds the highest temperatures associated with class B3, while Gerasimovič finds class B5 considerably hotter than B1. Temperatures based on King's colour indices show similar effects, and the question naturally arises whether the colour temperatures or those derived from ionization theory are more reliable in forming an estimate of the actual mean surface temperature of the star.

This question gains additional interest in view of the still more marked divergence between colour temperatures and ionization temperatures for O-type stars. The ionization temperatures, based on absorption-line intensities, range from 25,000° to 35,000°. The colour temperature for the absorption O's, as determined by Gerasimovič for some 12 stars, has the low average value of 8600°—considerably lower than that derived from the Bo to B5 stars.

Gerasimovič has also determined colour temperatures for a number of Wolf-Rayet stars. His data for these stars were less adequate than for the absorption O's, and the presence of emission bands naturally renders more difficult a precise evaluation of the energy relations in the continuous spectrum. It should be pointed out, however, that a number of the Wolf-Rayet stars selected by him for measurement showed comparatively weak bands, with considerable stretches of the continuous spectrum free from emission, so that there seems reason to suppose that colour temperatures derived from them should be of at least the right order of magnitude. The

* Payne, *Stellar Atmospheres*, p. 139, 1925.

† Hertzsprung, *Ann. Leiden Obs.*, **14**, No. 1, 1922; King, *H.A.*, **85**, No. 10, 1928; Payne, *Harv. Mon.*, No. 3, p. 92, 1930.

‡ Wilsing, *Pub. Obs. Potsdam*, No. 74, 1919; Greaves, Davidson and Martin, *M.N.*, **87**, 352, 1927; Sampson, *ibid.*, **90**, 636, 1930; Gerasimovič, *H.C.*, 339, 1929.

results showed a mean temperature of 17,000°, much higher than that of the absorption O's but less than would be expected from the theory of ionization as applied to absorption O's.

The almost complete lack of correlation between ionization temperatures and colour temperatures for early-type stars suggests that there is present some effect which operates to render invalid one or other of these methods of estimating stellar temperature. In the circumstances it seems more natural to suspect the colour temperatures, since it is difficult to accept the idea that the high degree of ionization existing in the atmospheres of O- and early B-type stars can be reconciled with temperatures of from 10,000° to 20,000°. In any case it is highly desirable to have some method of attack on the problem which will provide an independent determination of the quantities involved. The possibility of such an independent method is suggested in this paper. It consists in the application of the theory of nebular luminosity developed by Zanstra * to a determination of the temperatures of certain Wolf-Rayet stars and of the star P Cygni which falls in the early B class. Since considerable interest attaches to the temperatures of novæ, a determination of the temperature of Nova Aquilæ in its later stages has also been made.

General Principles of Zanstra's Theory

Zanstra's theory of nebular luminosity and its application to the temperatures of planetary nuclei has been described in detail in the publications quoted above, so that a brief outline of the underlying assumptions and a statement of the fundamental equations will suffice for this paper. He postulates a central star of high temperature surrounded by nebulous gases : our familiar picture of a planetary nebula. The star is considered to radiate as a black body of temperature T. The ultra-violet starlight beyond the head of the principal series of a given atom (the ionizing frequency) is assumed to be absorbed by the nebulous gases surrounding the star, each quantum absorbed causing the ejection of a photo-electron. On the assumption that absorption is complete, the number of quanta of ultra-violet light emitted beyond the ionizing frequency will be equal to the number of photo-electric ionizations. If the nebula is in a steady state, the number of ionizations which occur per second will be equal to the number of recombinations. It then becomes possible to equate the number of recombinations occurring per second to the number of quanta of ultra-violet starlight emitted per second beyond the ionizing frequency of the atom concerned.

The number of quanta emitted per second in the available region of the ultra-violet may be evaluated as a function of the ionizing frequency and the temperature of the central star. Zanstra has shown that it is represented by the following expression :

$$N_{ul} = \frac{8\pi^2 R^2 k^3 T^3}{c^2 h^3} \int_{x_0}^{\infty} \frac{x^2}{e^x - 1} dx. \qquad (1)$$

* *Ap. J.*, 65, 50, 1927; *Pub. D.A.O.*, 4, No. 15, 1930; *Zs. f. Astrophysik*, 2, 1, 1931.

$x=\frac{h\nu}{kT}$ and $x_0=\frac{h\nu_0}{kT}$, where ν_0 is the ionizing frequency, and the universal constants have their usual values, $h=6{\cdot}55\times10^{-27}$ and $k=1{\cdot}371\times10^{-16}$.

If the assumption be made that captures of electrons take place mainly in the outer orbits of the atom, then an approximate estimate of the number of recombinations occurring per second may be made from a consideration of the intensities of observed nebular lines relative to the continuous spectrum in the region in which they occur. Zanstra has shown that the number of quanta represented by a single line is given by the expression

$$N_p=\frac{8\pi^2R^2k^3T^3}{c^2h^3}\frac{x^3}{e^x-1}A_\nu,$$

where $x=\frac{h\nu}{kT}$ and A_ν is an observational quantity depending on the strength of the line. Since a series of lines is ordinarily involved, the approximate expression for the number of quanta becomes

$$\sum N_p=\frac{8\pi^2R^2k^3T^3}{c^2h^3}\sum\frac{x^3}{e^x-1}A_\nu, \tag{2}$$

the summation including as many lines in a given series as can be observed, and the quantity A_ν being determined separately for each line.

The observational quantity A_ν is defined by the expression

$$A_\nu=\frac{L_p}{\nu\frac{\partial L_s}{\partial\nu}}, \tag{3}$$

where L_p is the total luminosity of a single monochromatic nebular image and $\frac{\partial L_s}{\partial\nu}$ is the luminosity per frequency unit of the continuous spectrum at the same frequency as that of the emission line.

Setting (1) equal to (2), the equation for temperature determination becomes

$$\int_{x_0}^{\infty}\frac{x^2}{e^x-1}dx=\sum\frac{x^3}{e^x-1}A_\nu. \tag{4}$$

The equation is solved by trial and that value of T for which both sides are equal represents the determined temperature of the star.

Application to Emission of Wolf-Rayet Type

The theory of Wolf-Rayet emission recently developed by the writer * suggests a central star of high temperature from which atoms are being continually ejected with high velocity into space. The star is thus surrounded by nebulous gases consisting of outward-moving atoms, and it is in these nebulous gases that the emission spectra of Wolf-Rayet stars are believed to originate. The width of the bands is considered to be a Doppler effect due to the high velocities of emitting atoms in the line of sight.

* *M.N.*, **90**, 202, 1929; *Pub. D.A.O.*, **4**, No. 17, 1930.

It was suggested by the writer in previous publications * that the mechanism of emission in Wolf-Rayet stars was probably similar to that operating in planetary nebulæ. The essential differences in conditions are (*a*) the effective extension of the gases surrounding a Wolf-Rayet star is probably very much smaller than in the case of a planetary nebula ; and (*b*) the atoms in the envelopes of planetaries do not show the large outward velocities of those ejected from Wolf-Rayet stars. There does not seem to be any sound reason, however, why these differences should seriously affect the mechanism of emission. The assumption has accordingly been made that the agency responsible for the appearance of Wolf-Rayet bands is ionization and recombination operating in the same manner as the mechanism proposed by Zanstra for planetary nebulæ.

On this assumption, temperatures have been derived for the two Wolf-Rayet stars, H.D. 191765 and H.D. 192163. Temperatures have also been computed for Nova Aquilæ and for the star P Cygni, since they are believed to be similar to Wolf-Rayet stars in so far as the mechanism producing their emission lines is concerned.

The Observational Quantity A_ν

The observational quantity A_ν as used by Zanstra for the temperatures of planetary nuclei is defined by equation (3), which is repeated here for convenience.

$$A_\nu = \frac{L_p}{\nu \frac{\partial L_s}{\partial \nu}}. \qquad (3)$$

Zanstra's spectra for planetary nebulæ were taken with a slitless spectrograph, and the quantity L_p represents the total integrated intensity of the nebular image as measured spectrophotometrically. The quantity $\frac{\partial L_s}{\partial \nu}$ is the intensity per unit frequency of the continuous spectrum measured in the same units and at the same wave-length as the nebular image.

The measurements on which the present determinations of temperature are based have been made with slit spectra. Since Wolf-Rayet stars do not show observable discs, the assumption seems justified that the intensities of the bands as measured on these spectra represent their true intensities relative to the continuous spectrum. The same should be true of P Cygni and Nova Aquilæ at the time the spectra were taken.

If the width of the spectrum be w, and I_b be the intensity of the band at any wave-length, then Zanstra's quantity L_p is represented for Wolf-Rayet stars by the integral

$$w \int_{\lambda_1}^{\lambda_2} I_b d\lambda,$$

where λ_1 and λ_2 represent the wave-lengths of the extreme edges of the band.

* *M.N.*, 90, 207, 1929 ; *Pub. D.A.O.*, 4, No. 17, 294, 1930.

Since $\nu = \frac{c}{\lambda}$, the quantity $\nu \frac{\partial L_s}{\partial \nu}$ becomes $-\lambda \frac{\partial L_s}{\partial \lambda}$. The intensity of the continuous spectrum per wave-length unit is given by the equation

$$\frac{\partial L_s}{\partial \lambda} = wI_s, \tag{5}$$

where I_s is the intensity of the continuous spectrum at the mean wave-length of the band and w is the width of the spectrum. Making the necessary substitutions, the quantity A_v then becomes

$$A_v = \frac{\int_{\lambda_1}^{\lambda_2} I_b d\lambda}{I_s \lambda}. \tag{6}$$

The quantity in the numerator is evaluated by measuring the area under the plotted contour of the band. I_s is found by measuring the intensity of the continuous spectrum on either side of the band and taking the mean as representing the intensity at the centre.

Numerical Data and Results

For the two Wolf-Rayet stars, H.D. 191765 and H.D. 192163, as well as for Nova Aquilæ, the temperature determinations are based on measures of the single band, 4686 of *He* II, as this is the first line of a series the remaining members of which are in the ultra-violet region and cannot be observed. For P Cygni the temperature has been determined both from the Balmer series of hydrogen and from the "5876" series of *He* I. In the Balmer series, H_α, H_β and H_γ have been measured. For *He* I the first two members of the series, 5875·62, 4471·48, have been used.

The methods of spectrophotometry employed for deriving intensities from most of the plates secured in the present investigation have already been described.* A calibration strip was impressed on each plate by means of an optical wedge and an auxiliary spectrograph. The photographic densities were measured with the aid of a direct-reading photo-electric microphotometer, while the translation into intensities was done in the usual way by means of a characteristic curve connecting density and log intensity. Two of the plates of P Cygni in the visible region were secured before the optical wedge was available, and these were calibrated with the aid of a rotating step sector placed in front of the slit of a grating spectrograph.

The plates of Nova Aquilæ were uncalibrated, and intensity measurements were obtained from them by means of a method due to Hogg.† This method consists in employing the theoretical intensity ratios of iron arc multiplets in conjunction with density measurements of the same lines in the iron comparison spectrum impressed on each plate at the time it was taken. In this way a calibration curve could be constructed and used in

* *M.N.*, 92, 196, 1932. † *H.C.*, p. 337, 1929.

the usual way to determine relative intensities in the star spectrum. The multiplets used were the $^7F - {}^7D$ group between 4859 and 5000, and the $^5P - {}^5D$ multiplet between 4430 and 4530. It was assumed that the properties of the plate were the same at the two wave-lengths, and a mean curve constructed from the two multiplets was employed.

The observational data for the two Wolf-Rayet stars, H.D. 191765 and H.D. 192163, is reasonably extensive, as a considerable number of plates (10 for each star) had been secured for the purpose of accurately determining the contours of the bands. A graph illustrating the results of measures of the band 4686 in the star H.D. 192163 has already been published * as an illustration for a paper describing the photo-electric microphotometer in use at this observatory. The total integrated intensity of the band was derived by measuring the area under this graph. A graph of the distribution of intensity in the 4686 band of H.D. 191765 has also been published,† though the measures were made on an earlier series of plates and are entitled to less weight than those used in the present paper. However, the difference between the two sets of measures is small and the published graph gives a good idea of the form and intensity of the band. It is considered that the probable errors of measurement for these two stars are about 5 per cent., though a considerably larger error than this may be introduced by the difficulty of accurately estimating the intensity of the continuous spectrum in the centre of the band.

For Nova Aquilæ four plates were measured. They were taken on the following dates : 1918 August 22, September 13, October 6 and October 13. These particular plates were selected because the densities of the band and of the continuous spectrum were such as to make it possible to use the curves constructed from the iron lines without extrapolation—a condition somewhat difficult to attain in uncalibrated spectra. It is practically certain that the intensity of the band 4686 relative to that of the continuous spectrum was increasing during this interval, so that the temperature derived must be regarded as a mean value over the period concerned. An attempt has been made to estimate the probable errors of the intensity determinations by the use of calibrated spectra obtained in routine observations of stellar spectra. For this purpose ten plates were selected entirely at random. For each plate two (density)/(log intensity) curves were constructed, one using the usual calibration strip and the other using the two iron multiplets mentioned above. When the slopes of different curves were compared, it was found that in four cases the slope of the curve constructed from the measured iron lines was greater than that constructed from the calibration strip ; in three cases it was less, and in three cases there was no clearly distinguishable difference. The measurements suggested that maximum errors of the order of 50 per cent. might be expected for individual plates, but where two or more plates were used the errors should be considerably less. A probable error of 30 per cent. for the final values of the average intensity of 4686 in Nova Aquilæ during the interval over which the spectra were taken would probably be a reasonable estimate.

* *M.N.*, **92**, 196, 1932. † *Ibid.*, **91**, 966, 1931.

The observational data for P Cygni consisted of four spectrograms taken with three-prism dispersion, two in the ordinary region of observation, 4000–5000 A., and two in the region 5100–6700. The two plates measured in the ordinary region were calibrated in the usual way. The measures showed good agreement, and the probable errors in the intensities should not be more than 15 per cent. The plates in the longer wave-length region were calibrated with a rotating sector, and unfortunately the intensities of 5875 and H_α on one of the plates were so great as to fall outside the range of the calibration curve. It has been the writer's experience that very little weight can be attached to measures of intensity derived from a curve extrapolated in the region of high densities. Accordingly, only the measures from the less dense plate have been used. Measures of the same lines on the denser plate using the extrapolated curves gave very much larger values, and this at least suggests that the values used for the temperature determinations may be too small.

It is, of course, well known that the use of a rotating sector as a means of calibration for photographic plates may lead to considerable errors. A series of measures for the purpose of testing this point has been made by the writer in the ordinary photographic region. These measures showed that very little error was involved for densities falling on the straight-line portion of the characteristic curve. For very high or very low densities, however, the errors were as high as 30 per cent. The writer considers that 50 per cent. is a fair estimate of the probable errors of the values of the total integrated intensities of the lines 5875 and 6563 in P Cygni.

Final results of the measures for all four stars are given in Table I.

TABLE I

Star	λ	$\int_{\lambda_1}^{\lambda_2} I_b d\lambda$	I_s	A_ν
H.D. 191765	4686	345	1·5	0·0491
H.D. 192163	4686	258	1·5	0·0367
Nova Aquilæ	4686	147	1·0	0·0209
P Cygni ...	6563	91·0	1·0	0·0139
	4861	12·7	1·0	0·0026
	4340	4·8	1·0	0·0011
	5875	12·4	1·0	0·0021
	4471	2·0	1·0	0·00045

The values of A_ν given in the fifth column of Table I are used in calculating the actual temperatures of the stars. The temperatures are derived by the solution of equation (4) repeated below:

$$\int_{x_0}^{\infty} \frac{x^2}{e^x - 1} = \sum \frac{x^3}{e^x - 1} A_\nu. \qquad (4)$$

The methods of evaluating the integral and solving the equation are discussed by Zanstra in considerable detail, and will not be touched on here. For the stars in which 4686 is used, the quantity $\sum \frac{x^3}{e^x - 1} A_\nu$ is evaluated on the assumption that the quanta appearing in the 4686 band represent half the total number of ultra-violet quanta originally absorbed. For the lines of hydrogen and helium in P Cygni, it is assumed that the number of quanta appearing in observed lines is equal to the number of quanta originally absorbed. These assumptions are only approximate, particularly in view of the fact that no account is taken of overlapping regions of absorption. It is hoped, however, that they will lead to results of the correct order of magnitude. The nature of the uncertainty involved will be discussed in a later paragraph.

Actual values for the temperatures are given in Table II. In the first column of the table is contained the designation of the star. In the second are the wave-lengths of the bands whose intensities have been determined. The third column gives the temperatures computed on the assumptions mentioned above. In the fourth column are given the temperatures computed on the assumption that the number of quanta appearing in the 4686 line represent the total number of quanta originally absorbed. In this column are also given temperatures for P Cygni computed on the assumption that all the values of A_ν in Table I should be divided by 2.

Table II

Star	λ	T	T_{min}
H.D. 191765	4686	71,000°	65,500°
H.D. 192163	4686	69,000°	64,000°
Nova Aquilæ	4686	64,500°	60,500°
P Cygni ...	6563	18,000°	16,000°
	4861		
	4340		
	5875	25,000°	23,000°
	4471		

Validity of the Results

In discussing the validity of the results it should be emphasized at the beginning that the theory of ionization and recombination gives only lower limits for the temperatures of the stars to which it is applied. This means that if the theory is valid in its main features, the actual temperatures of the stars should be at least as high as the values given in the third column of Table II. Since the values given in this table are conspicuously higher than those obtained for stars of similar type by other methods, it seems desirable

to review the assumptions involved, in an endeavour to detect possible flaws in the argument.

The first assumption, that the star radiates approximately as a black body, seems inherently probable physically and has formed the basis of most determinations of stellar temperature that have so far been made. Evidence has been adduced by Gerasimovič * from spectrophotometric data which suggests that there are large departures from black-body radiation for the O- and B-type stars. It should be pointed out, however, that his low colour temperatures for early-type stars could be explained as an effect of selective absorption of light in space, although he considers his "superexcitation" hypothesis more probable. Further, the stellar region of observation represents a comparatively small part of the whole spectrum, and a deviation from black-body laws in this region may not necessarily mean that the energy curve for the spectrum as a whole has no similarity to that of a black body. Zanstra's theory effectively compares the intensity of radiation in the far ultra-violet with that of the ordinary region taken more or less as a whole. For the purpose of determining the true average surface temperature of a very hot star, this comparison seems likely to be more significant physically than a determination of the gradient of the energy curve in the ordinary region of observation.

The assumption that the mechanism of ionization and recombination is operating to produce the emission bands of Wolf-Rayet stars and novæ derives its probability mainly from the apparent impossibility of finding any other mechanism efficient enough to produce the observed effects. Calculations based on the mechanism of line absorption and re-emission (the only very obvious alternative) indicate that even on the most favourable assumptions there would not be sufficient energy available to produce the very intense bands of Wolf-Rayet stars and novæ.

The assumption that absorption of ultra-violet starlight is complete, is almost undoubtedly not fulfilled, especially since there will be regions of absorption for different elements which will overlap. Failure of this assumption will lead to too low a value for the stellar temperature, however, so that the claim of the theory to derive minimum temperatures will not be affected. Failure of the assumption that captures are made mainly in the outer orbits of the atom will also lead to too low a value for the stellar temperature.

Some idea of the effect of observational errors on the final results may be gained by a comparison of columns 3 and 4 of Table II. Dividing the number of quanta by 2 lowers the temperatures of the Wolf-Rayet stars and Nova Aquilæ by 5000° but leaves the general order of magnitude of the results unchanged. The percentage change in the temperature of P Cygni is of about the same order, and indicates that any observational errors which may be introduced are unlikely to have any appreciable effect on the computed temperatures.

* *H.C.*, No. 339, 1929.

General Discussion

(a) *The Wolf-Rayet Stars.*—The results of Table II show that Wolf-Rayet stars with a strong band of *He* II 4686 have temperatures of 70,000° or over. This result is in agreement with the general order of temperatures derived for planetary nuclei by the same method, and should, presumably, be entitled to equal weight. The two Wolf-Rayet stars of Table II show spectra of type OW6, in which 4686 appears at approximately its maximum intensity. It is hoped eventually to extend the measurements to other spectral types. In the meantime it is possible, by a calculation of the values of the quantity A_ν corresponding to different temperatures, to draw certain general conclusions concerning the group of Wolf-Rayet stars taken as a whole. Table III gives approximate values of A_ν corresponding to a range

Table III

T	A_ν
80,000°	0·1287
75,000°	0·0765
70,000°	0·0463
65,000°	0·0222
60,000°	0·0097
55,000°	0·0039
50,000°	0·0011

of temperatures from 50,000° to 80,000°. It may be seen from this table that the value of A_ν corresponding to a temperature of 70,000° is 0·0463. For a star of temperature 50,000°, $A_\nu = 0{\cdot}0011$—approximately $\frac{1}{4}\frac{1}{0}$ of the value for 70,000°. It is practically certain that the range of variation in the intensity of this wave-length for ordinary Wolf-Rayet stars is less than this. A value of from $\frac{1}{1}\frac{1}{0}$ to $\frac{1}{2}\frac{1}{0}$ would probably more nearly represent the actual facts. This means that the Wolf-Rayet stars as a class probably have temperatures ranging from a lower limit of 55,000° up to possibly 75,000°. There does not seem anything unreasonable about the upper limit of 75,000°, particularly since the class OW6, to which the two stars considered in this paper belong, is not the earliest type of Wolf-Rayet spectrum known. The class OW5 exhibits a greater degree of ionization in its spectrum, and though 4686 is perhaps somewhat weaker, this is very probably because most of the helium is doubly ionized.

The results for the temperatures of the Wolf-Rayet stars have an important bearing on the temperatures of the absorption O's because a number of stars are known which have well-marked absorption spectra in addition to their Wolf-Rayet emission bands. Examples of such stars are H.D. 193793, which has an absorption spectrum O5 and a Wolf-Rayet spectrum OW8, and the star H.D. 211853, with an absorption spectrum O7 and a Wolf-Rayet spectrum OW7. The fact that the Wolf-Rayet bands are present in

considerable strength would appear to indicate that the temperatures of these stars should fall between the limits 55,000° and 75,000° suggested above. Since there is nothing unusual about the absorption spectra of these stars there is reason for believing that the temperatures of absorption O's are of the same general order as those of Wolf-Rayet stars. It may be, in fact, that the distinction between the absorption O's and the Wolf-Rayet stars is not one of temperature but has to do with some other physical property, such as the value of gravity at the stellar surface.

Finally, it remains to mention the enormous difference between the values derived in this paper for the temperatures of the Wolf-Rayet stars and the values obtained as a result of spectrophotometric observations of the continuous spectrum in the stellar region. If the present results be accepted as valid, the conclusion is obvious that little reliance can be placed on colour-temperature determinations for early-type stars.

(b) *Nova Aquilæ.*—The temperature of Nova Aquilæ comes out somewhat lower than that of the Wolf-Rayet stars but is of the same general order. As has already been mentioned, the temperature was probably changing during the period covered by the spectrograms which have been measured, and so the value given here (65,000°) must be regarded as a mean value over the period concerned. The value of 65,000° is rather higher than we have been accustomed to associate with novæ, but the appearance of 4686 in considerable strength and the presence of the nebular lines N_1 and N_2 are both indications of very high temperature. Even a very weak band of 4686, according to Zanstra's theory, implies a temperature of 55,000° or over. As a consequence, it seems probable that most of the novæ observed of recent years have attained temperatures of 60,000° or more. Nova Cygni (1920) is probably an exception, as the writer has not been able to distinguish 4686 on any of the Victoria plates and the spectrum in general indicates a lower degree of ionization than that of Nova Aquilæ.

(c) *P Cygni.*—Two different values of the temperature have been derived for P Cygni, one from the lines of hydrogen and one from those of *He* I. Disregarding the possibility of observational error affecting the results, the higher temperature must be accepted as more nearly correct, since the theory gives minimum values only. In the present instance, the spectrophotometric results are almost certainly better for the helium lines than for those of hydrogen, owing to the difficulty of accurately measuring the very strong H_α line. Consequently, the value of 25,000° derived from the lines 5876 and 4471 of *He* I has been taken as representing the lower limit of the temperature of P Cygni. As in the case of the Wolf-Rayet stars, this value is in marked disagreement with colour-temperature data. The colour temperature derived for this star by Gerasimovič * is 6000°. It is classified as B1, and as there appears to be no trace of 4686 in its spectrum, its temperature would be expected to come out considerably lower than that of the Wolf-Rayet stars. The temperature of class B1, on the basis of ionization theory, is some 18,000°, a value not greatly different as regards order of magnitude from the values of Table II.

* *H.C.*, No. 339, 1929.

Summary and Conclusion

The temperatures of the Wolf-Rayet stars, H.D. 191765 and H.D. 192163, have been calculated on the assumption that their emission spectra are produced by the mechanism of ionization and recombination suggested by H. Zanstra. The validity of the theory is discussed, and reasons are advanced for placing confidence in the results.

The temperatures of the two Wolf-Rayet stars come out approximately 70,000°, and it is suggested that the temperatures of the class as a whole range from approximately 55,000° to 75,000°. There are reasons for believing that the absorption O's have temperatures of the same general order. Attention is directed to the marked divergence between the colour temperatures and those derived in this paper.

The temperature of Nova Aquilæ is computed to be in the neighbourhood of 65,000°. It is suggested the majority of novæ attain temperatures of this order.

The temperature of P Cygni has been calculated to be 25,000°. Again there is a very large difference between the colour temperature (6000°) and the value derived from the theory of ionization and recombination.

In conclusion I wish to thank Dr. F. S. Hogg, of the staff of this observatory, for kindly checking the calculations and for helpful discussions during the course of the work. I am also indebted to Mr. S. N. Hill for assistance in the measurement and reduction of the plates.

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Victoria, B.C., Canada:
1932 *March* 3.

THE GRAPHICAL COMPUTATION OF THE RADIAL COMPONENT OF THE SOLAR MOTION.

A. F. Dufton, M.A., D.I.C.

(Communicated by E. A. Milne)

1. In volume vi, No. 4, of the *Publications of the Dominion Astrophysical Observatory, Victoria, B.C.*, Messrs. J. A. Pearce and S. N. Hill describe a convenient form of a chart which has been in constant use for eliminating the solar motion from the observed radial velocities of stars. In this elegant solar motion graph, contours drawn on squared paper provide a graphical solution of the equation

$$V = 20[\sin\delta \sin 28° + \cos\delta \cos 28° \cos(271° - \alpha)], \qquad (1)$$

where V is the projection, in the line of sight of the star whose co-ordinates are (α, δ), of the solar motion of 20 km./sec. towards the apex (271°, +28°).

2. Equation (1) can be expressed in the form

$$\begin{vmatrix} 1 & 0 & -V/20 \\ 0 & 1 & -\cos(271^\circ - \alpha) \\ 1 & -\cos 28^\circ \cos\delta & -\sin 28^\circ \sin\delta \end{vmatrix} = 0,$$

which is the eliminant of the three equations,

$$u \qquad -V/20 \qquad = 0,$$
$$v \qquad -\cos(271^\circ - \alpha) = 0$$

and

$$u \quad -v\cos 28^\circ \cos\delta \quad -\sin 28^\circ \sin\delta \quad = 0,$$

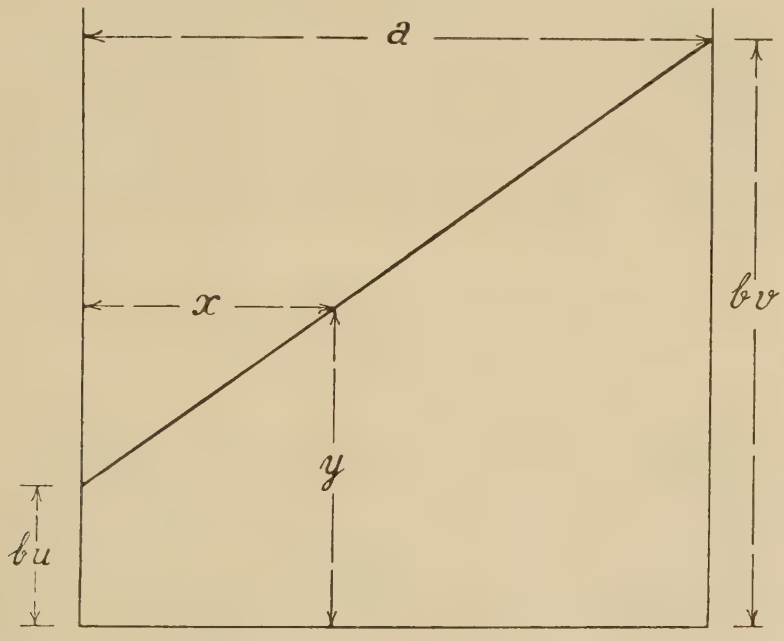


FIG. 1.

and the condition of collinearity of the points which they define in parallel co-ordinates, u and v.

The relation between parallel co-ordinates and cartesian co-ordinates is seen in fig. 1. From similar triangles it is clear that

$$x/a = (y - bu)/(bv - bu);$$

that is,

$$u(1 - x/a) + vx/a - y/b = 0,$$

an equation of the first degree in u and v defining the point (x, y).

Transferred to cartesian co-ordinates, by comparison with this equation, the first two points are

$$x = 0, \qquad y = bV/20$$

and

$$x = a, \qquad y = b\cos(271^\circ - \alpha),$$

and define two parallel straight scales, while the third point is given by

$$\frac{1 - x/a}{1} = \frac{x/a}{-\cos 28^\circ \cos \delta} = \frac{y/b}{\sin 28^\circ \sin \delta}, \qquad (2)$$

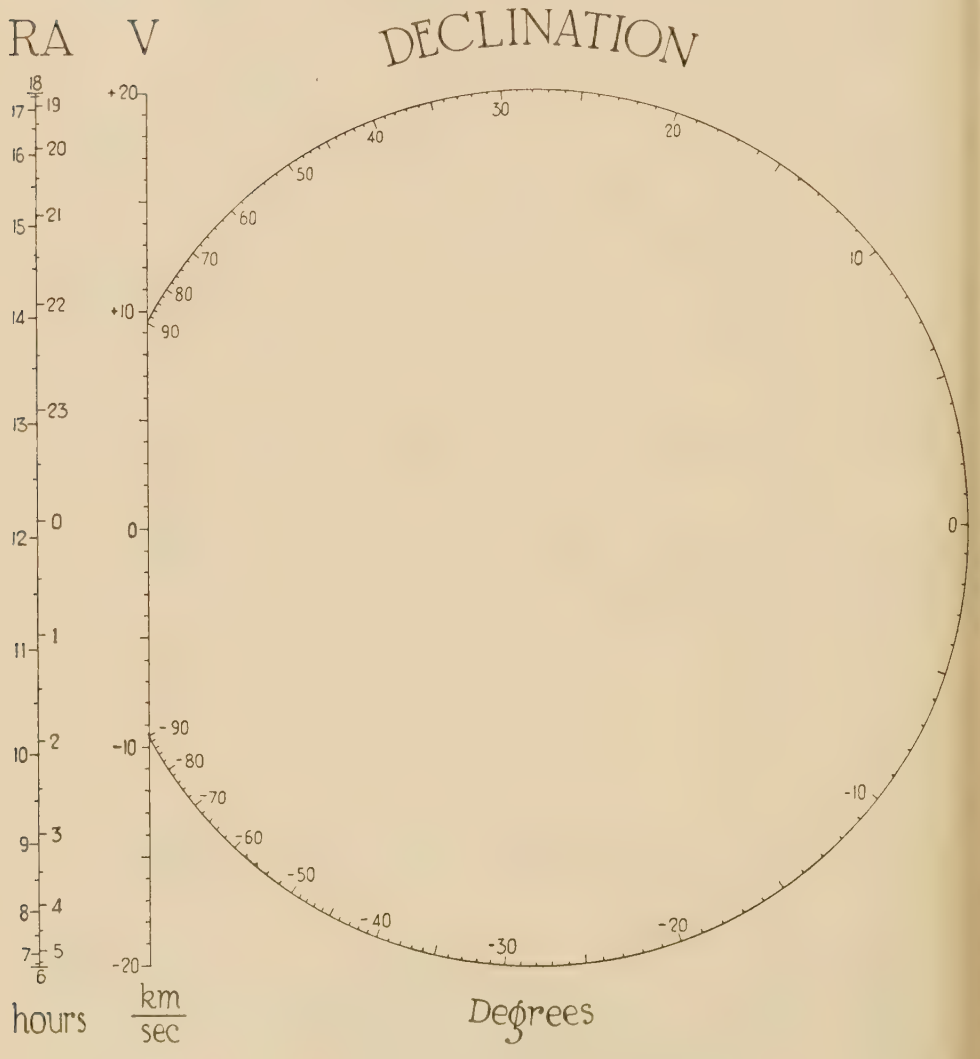


FIG. 2.—*Alignment Diagram for Solar Motion.*

the locus being the conic

$$\left(\frac{x}{a \cos 28^\circ}\right)^2 + \left(\frac{y}{b \sin 28^\circ}\right)^2 = \left(\frac{a - x}{a}\right)^2,$$

which reduces to the circle

$$(x - b \cos 28^\circ)^2 + y^2 = b^2$$

if a be made equal to $-b \sin 28^\circ \tan 28^\circ$.

3. Equation (1), therefore, is solved by the simple alignment diagram shown in fig. 2. This is set out by choosing a suitable unit b for the scales of velocity and longitude and drawing them at a distance apart equal to $b \sin 28° \tan 28°$; the longitude scale is conveniently graduated in hours of right ascension. The scale of declination is a circle of radius b with its centre at a distance $b \cos 28°$ from the velocity scale : it is readily projected, since, from (2),

$$y/x = -(b/a) \tan \delta \tan 28°$$
$$= \tan \delta / \sin 28°.$$

In using the diagram, a straight edge, or preferably a straight line ruled on the under surface of a sheet of celluloid, is laid across so as to pass through the two points representing the co-ordinates of the star, say R.A. $13^h\ 20^m$, Dec. $-10°$. The point on the scale of velocity through which it passes indicates $+4{\cdot}0$ km./sec., the radial component of the solar motion.

4. It will be appreciated that such an alignment diagram is somewhat simpler to construct than the intersection diagram of Messrs. Pearce and Hill. It may, moreover, be more convenient to use, as there is no tracing back along a line to read its graduation and interpolation by eye is only necessary upon simple scales.

THE REINMUTH OBJECT.

Rev. M. Davidson, B.A., D.Sc.

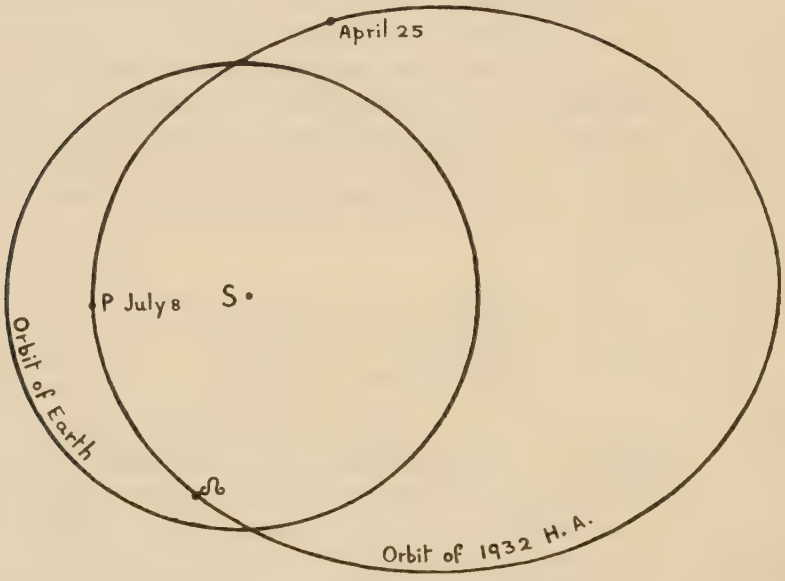
This object was discovered by Dr. Reinmuth at Königstuhl on April $24^d\ 23^h\ 54^m{\cdot}4$. Its magnitude at the time of discovery was stated to be 12·5, but Dr. W. H. Steavenson, who observed it on a number of occasions, gave its magnitude as varying between 11·5 and 12. Drs. Whipple and Cunningham computed * a parabolic orbit and gave an ephemeris up to June 2, but the predicted positions did not fit in very closely with later observations. Dr. Steavenson described the object as stellar, and was of opinion that it was a minor planet. Dr. Stracke also computed † an orbit, and found that its periodic time was about $1^y{\cdot}666$.

Using observations made at Königstuhl on April 24, May 1, and at Norwood by Dr. Steavenson on May 6, an attempt was made to compute a general orbit, but the usual methods failed. The same thing occurred with the Delporte object which, like the present one, was near opposition at the time of discovery. The method adopted finally was a modification of that which I described in *Memoirs of the B.A.A.*, vol. **30**, part 1, 1932. The periodic time was found to be 1·77006 years, but this was subject to some correction, because Dr. Steavenson's observation had been made under difficulties, and, as Dr. Steavenson stated, might possibly be in error by about 20″. On May $8^d\ 23^h\ 18^m{\cdot}2$ Dr. Steavenson got a more accurate observation, and this was used in conjunction with the Königstuhl results to recompute an orbit, the elements of which are given below.

* *Harvard Card*, 212. † *I.A.U. Circ.*, 374.

This object is unique amongst the asteroids as its orbit comes within that of the Earth. Its nearest approach to the Sun is 0·651 on July 8, and its aphelion distance is nearly 2·26. It moved south of the ecliptic on April 28, and will reach the ascending node on August 17; but as it will then be more than a unit from the Earth, and as the angle Earth-Sun-Object at the time is about 70°, observations of a transit will be impossible.

The near approach to the Earth produced considerable parallaxes. Dr. Crommelin used these parallaxes, as he also did in the case of the Delporte object, to determine the geocentric distances, and in both cases these agreed almost exactly with those found by other methods. As a general rule a close approach to the Earth renders the computation of an orbit more difficult



owing to the large corrections for parallax, which can be made only after the geocentric distances are found approximately. Dr. Crommelin reversed the process and found the geocentric distances from the parallax.

The object must be very small—probably only a mile or two in diameter—and it will soon become difficult to observe. It is to be hoped that further observations will be made so that more accurate elements can be computed. In seven years it should return to nearly the same position relative to the Earth, and possibly may be found again. No account has been taken of the perturbations of the body by our planet; these are too large to be ignored completely, as it has remained close to the Earth for some time. A few of the approximate distances are given: the nearest approach will be on May 15, when it will be about $6\frac{1}{2}$ million miles from us.

It seems quite possible that there are other planets of a similar character with orbits coming inside that of the Earth, but they have escaped detection so far. Perhaps the discovery of Planet 1932 H.A., the designation given to the object, will assist in throwing some additional light on the past history

of our solar system. The diagram shows the orbits of the new planet and of the Earth.

Orbit of Planet 1932 H.A.

T	1932 July 8·0900	
ω	284° 47′ 10″	1932·0
Ω	35° 52′ 08″	
i	6° 13′ 47″	
e	0·552376	
n	2022″·09	
a	1·454803	
ρ	1ʸ·754714	

Distances from the Earth

May	5	0·110
	7	0·097
	9	0·086
	11	0·078
	13	0·072
	15	0·070
	17	0·072
	19	0·077
	21	0·085

Addendum

Dr. Stracke has computed * another orbit in which four observations have been used, and this is almost certainly nearer the true result than that given above. Further corrections will probably be made if more observations are secured.

Dr. Stracke's Orbit

T	1932 July 9·38353	
ω	284° 31′ 00″·8	1932·0
Ω	35° 56′ 15″·9	
i	5° 49′ 32″·3	
e	0·520908	
n	2164″·26	
a	1·390372	
ρ	1ʸ·639445	

Dr. Wyse and Miss Covey have computed † an orbit, using later observations than Dr. Stracke employed, and have found a period of 1ʸ·7975. I have also computed a later orbit, using an observation made at Milan on May 12. The period deduced in this case is about 7 days less than that given in my orbit above. It is possible that further observations will be secured in August, and better results will then be obtained. The elements of my latest orbit are as follows :—

T	1932 July 8·28037	
ω	284° 46′ 58″	1932·0
Ω	35° 51′ 00″	
i	6° 10′ 32″	
e	0·547721	
n	2043″·73	
a	1·444514	
ρ	1ʸ·736131	

* *I.A.U. Circ.*, 376. † *Harvard Card*, 218.

THE APPLICATION OF THE HOLLERITH TABULATING MACHINE TO BROWN'S TABLES OF THE MOON.

L. J. Comrie, M.A., Ph.D. (Plates 11–13.)

Introduction.—Positions of the Moon derived from Brown's *Tables of the Motion of the Moon* were first introduced into national ephemerides in 1923. In these *Tables* over 1400 periodic terms with coefficients as large as 0″·003 are tabulated. Some idea of the extent of the *Tables* may be gathered from the fact that they contain 180 separate tables and cover 660 pages. Although

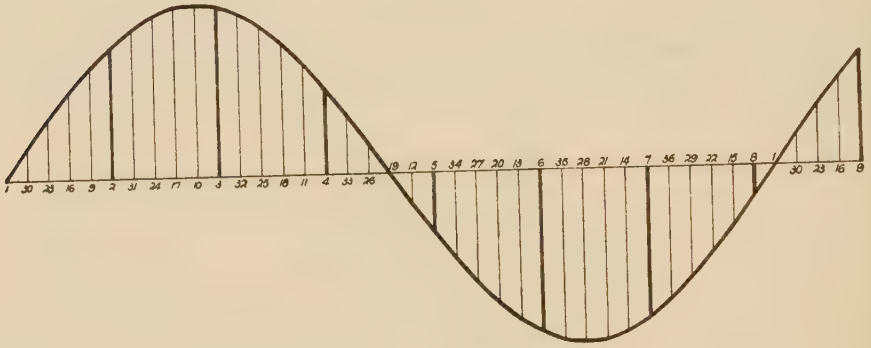


Fig. 1.

they are well arranged, the work of using them is laborious, and, before the advent of the Hollerith machine, represented the continuous work of two skilled computers. The mechanical methods that have been applied to certain portions of the work have eliminated much fatigue, increased tenfold the speed with which results can be obtained, and reduced the cost to one-quarter of its former amount.

The Tabulation of Harmonic Functions of the Time.—The inequalities arising in lunar theory are of the form $a \sin (b + ct)$, where a is the amplitude or coefficient, b is the phase angle at the moment when $t = 0$, and c is the movement of the argument in a unit of time. This expression is called a harmonic ; it represents a periodic term, of period $2\pi/c$. The *Tables* present values of these terms, and it is their selection and summation, or the synthesis of a series of harmonic terms, that constitutes the principal work of using the *Tables*. Where the periods of various terms are commensurable, for instance if they depend on $(b + ct)$, $2(b + ct)$, $3(b + ct)$, etc., they are combined into one table ; it is this power of combination that has enabled 1400 terms to be tabulated in less than 200 tables. The fundamental periods of individual tables are not commensurable, so that, although the entries from each table are used over and over again, combinations of entries from any one group of

tables do not recur. It is to be borne in mind that in preparing ephemerides the intervals of t are uniform, and that the period of each term is, in general, short when compared with the lifetime of the *Tables*.

Consider the tabulation of a term in which, for simplicity, $a = 1$, $b = 0°$, $c = 50°$, and the unit of t is 1 day. The values required on successive days are represented by the ordinates 1, 2, 3 . . . in fig. 1. If we superimpose the values for day 9 and subsequent days on the first period of the curve, it is seen that this is eventually divided into 36 equidistant parts, after which the sequence repeats itself. This immediately suggests the procedure actually adopted in tabulation, namely, that a complete period of the harmonic should be subdivided into an integral number of parts, so chosen that the motion in a day (or other unit of time) is an integral number of these parts. In the illustration 1 day $= \frac{50°}{360°} = \frac{5}{36}$ revolutions, so that the constructor of the tables would divide the period into 36 integral parts, the movement per day being 5 of these parts. The order of the final tabular entries would be the order in which they are used.

In this simple illustration the daily motion in revolutions is represented by an exact fraction, but this is not the case in general. Consider the term $-1''{\cdot}272 \sin 2L$ in the long-period nutation in longitude, where L is the Sun's mean longitude. The daily motion of $2L$ (at epoch 1950) is $1°{\cdot}97129469$, or, in revolutions of 360°, $0^{r}{\cdot}00547581 86$. This number of revolutions has to be represented by a proper fraction in its lowest terms. With the aid of a calculating machine one finds readily the figures in the accompanying table, the development being made in such a way that the residuals alternate in sign, each being numerically less than the preceding one.

d	r	r
182 =	1	−0·003401
183 =	1	+0·002075
365 =	2	−0·001326
548 =	3	+0·000749
913 =	5	−0·000578
1461 =	8	+0·000171
5296 =	29	−0·000065

Hence $\frac{8}{1461}$ or $\frac{29}{5296}$ revolutions are close approximations to the daily motion of $2L$. If we adopt the former for our purpose, a table may be constructed giving $1{\cdot}272 \sin \frac{360° \times k}{1461}$ where k varies from 0 to 1460. By rearranging the 1461 entries in this table in the order, 1, 9, 17 . . . 1449, 1457, 4, 12 . . . 1452, 1460, 7, 15 . . . 1447, 1455, 2, 10 . . . 1450, 1458, 5, 13 . . . 1453, 1461, 8, 16 . . . 1448, 1456, 3, 11 . . . 1451, 1459, 6, 14 . . . 1446, 1454, 1, 9, 17 . . . they will represent the values of the term required at consecutive intervals of 1 day. It is seen that entry No. 1 (corresponding to $k = 0$) is not used a second time until every other entry has been used once. The cycle of entries, when once started at the proper place, could be used indefinitely, except for the fact that $\frac{8}{1461}$ is not a perfect representation of the mean daily motion of $2L$, while even this mean motion is not constant. The entries are given permanent consecutive numbers representing the order in which they are used.

k	No. used above	$2L$	Order of use, or card number	Entry in units of 0″·001
		°		
0	1	0·000	1	0
1	2	0·246	549	− 5
2	3	0·493	1097	− 11
3	4	0·739	184	− 16
4	5	0·986	732	− 22
5	6	1·232	1280	− 27
6	7	1·478	367	− 33
7	8	1·725	915	− 38
8	9	1·971	2	− 44

Hence the final table runs thus :—

No.	Entry
1	0
2	− 44
3	− 87
4	− 131
5	− 175

Since a revolution has been divided into 1461 parts, the interval between these parts is 0°·2464. The value of $2L$ will, in general, lie between two of those used in constructing the table. Thus if, at some given date, $2L = 1°{\cdot}000$, the entry required (see the line $k = 4$ in above table) may be expressed as $732^c{\cdot}06$, since, as will be seen later, each tabular entry is represented by a card. Similarly the entry for 1°·200 would be $732^c{\cdot}87$, or $1280^c - 0^c{\cdot}13$. From the table in which the fraction $\frac{8}{1461}$ was developed, it is seen that the adopted value of $2L$ at the end of 1461 days is $0^r{\cdot}000171$ too small. Thus the true interval between consecutive days is, in units of $\frac{1}{1461}$ revolutions, 8·000171. The advance in card number corresponding to 8 units is 1 ; hence the card number advances 1·000171 per day, the integral part, by virtue of the final numbering of the cards in the actual order of use, denoting 8 units, and the decimal part denoting units of $\frac{1}{1461}$ revolutions. If F be the fraction of the card number required to represent the true entry at any date, it is easily seen that, at any initial time, F need not exceed $\pm 0{\cdot}5$. It increases by 0·000171 per day, or by 1 every sixteen years. Hence once every sixteen years there must be a discontinuity in the sequence of card numbers to change a value of F just greater than 0·5 to a value just numerically less than $-0{\cdot}5$. From the column "Order of use" it is seen that entries that are adjacent in numerical value, *e.g.* 732 and 1280, differ by 548 in their assigned numbers. Hence when F exceeds 0·5 an addition of 548 must be made to the card number to obtain the card number corresponding to the nearest tabular entry ; or, again, $732^c{\cdot}50 = 1280^c - 0^c{\cdot}50$.

The correction required to allow for the fraction or interpolating factor F is

$$-\frac{2\pi F \times 1''\cdot272 \cos 2L}{1461} = -0''\cdot0055F \cos 2L$$

and, since F need not exceed $\pm 0\cdot5$, this will not exceed $\pm 0''\cdot003$. In computing nutation all terms with coefficients less than $0''\cdot004$ are omitted, so that the omission of a correction or proportional part that may attain $\pm 0''\cdot003$ is comparable with omitting a term whose coefficient is $0''\cdot003$. The proportional parts omitted in the other nutation terms do not exceed $0''\cdot001$. It would, of course, have been possible to reduce the proportional parts of this term to a maximum of $0''\cdot001$ by adopting $\frac{29}{5296}$ revolutions as the daily motion of $2L$. But this would have involved the making of 5296 tabular entries, and the gain in accuracy would be inappreciable, as a value of the nutation to within $0''\cdot01$ is ample for the present needs of astronomy.

The single-entry tables (*i.e.* those that depend on a single argument) of Brown's *Tables* are arranged in the manner described. Hence the user ascertains first the tabular entry corresponding to his initial date; the entries for subsequent consecutive dates are simply subsequent consecutive entries in the table, with occasional discontinuities. These entries may be copied in order from each table, and the transcribed entries added. This becomes laborious, mainly because of the number of entries to be copied and added—about 120,000 a year, as positions are computed for each noon and midnight. Also the risk of errors in copying and adding half a million figures is not inconsiderable. In the process about to be described the copying is done once only, the addition is done mechanically, and the results of the addition are printed.

*Hollerith Cards and Punches.**—The basic unit in Hollerith equipment is a card $7\frac{3}{4} \times 3\frac{1}{4}$ inches in size, divided into 45 vertical columns,† as shown in fig. 2. Each column contains 12 vertical positions, named, from top to bottom, Y, X, 0, 1 . . . 9.‡ A group of several columns constitutes a field, and in each field a number (within the capacity of the field) may be entered by punching a hole in each column. The key-punch is shown in fig. 3. The card is carried by a movable carriage, each key depression driving a knife through the card, and releasing the carriage so that it moves to the next column. Electrical punches are also available; in these the depression of any key merely actuates a magnet, which drives the appropriate knife through the card. In the electric duplicating punch (fig. 4) a completely punched master-card, if placed in the upper frame, will be automatically copied, at the rate of 10 holes a second, on a card in the lower part of the machine. If any columns of the master-card are unpunched, the machine will naturally

* A description of the Hollerith machines, containing more mechanical details, has been given by the writer in an article, "The Hollerith and Powers Tabulating Machines," in *Office Machinery Users' Association Transactions*, 1929–30.

† Machines using a card of the same size, but divided into 80 columns, are also in use.

‡ The letters Y and X are not printed on the card.

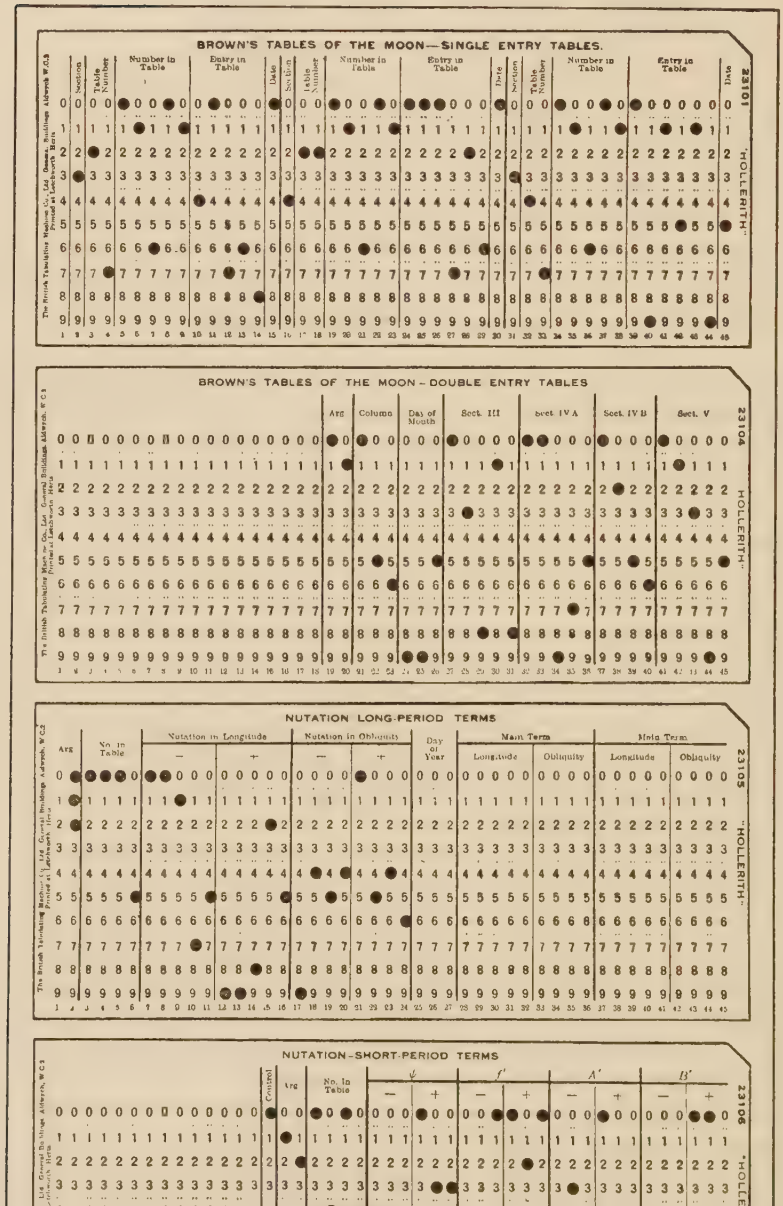

BROWN'S TABLES OF THE MOON—SINGLE ENTRY TABLES.
BROWN'S TABLES OF THE MOON—DOUBLE ENTRY TABLES
NUTATION LONG-PERIOD TERMS
NUTATION—SHORT-PERIOD TERMS
HOLLERITH

FIG. 2.—*Punched Cards.*

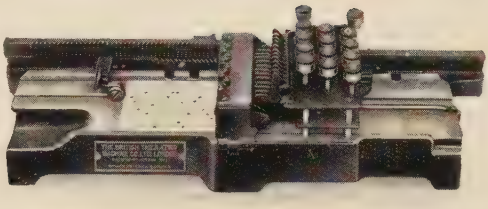


FIG. 3.—*Hand Key-punch.*

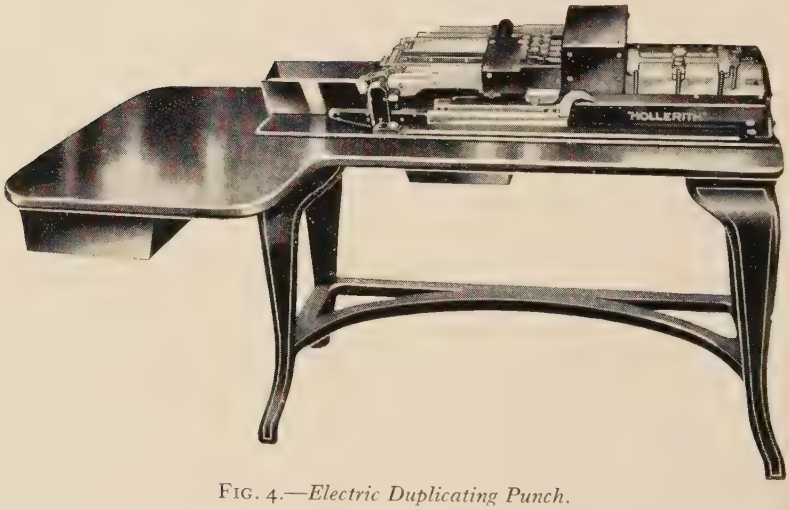


FIG. 4.—*Electric Duplicating Punch.*

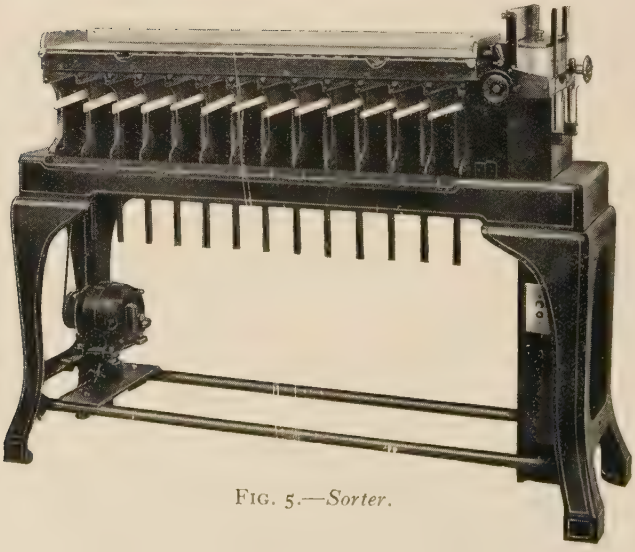


FIG. 5.—*Sorter.*

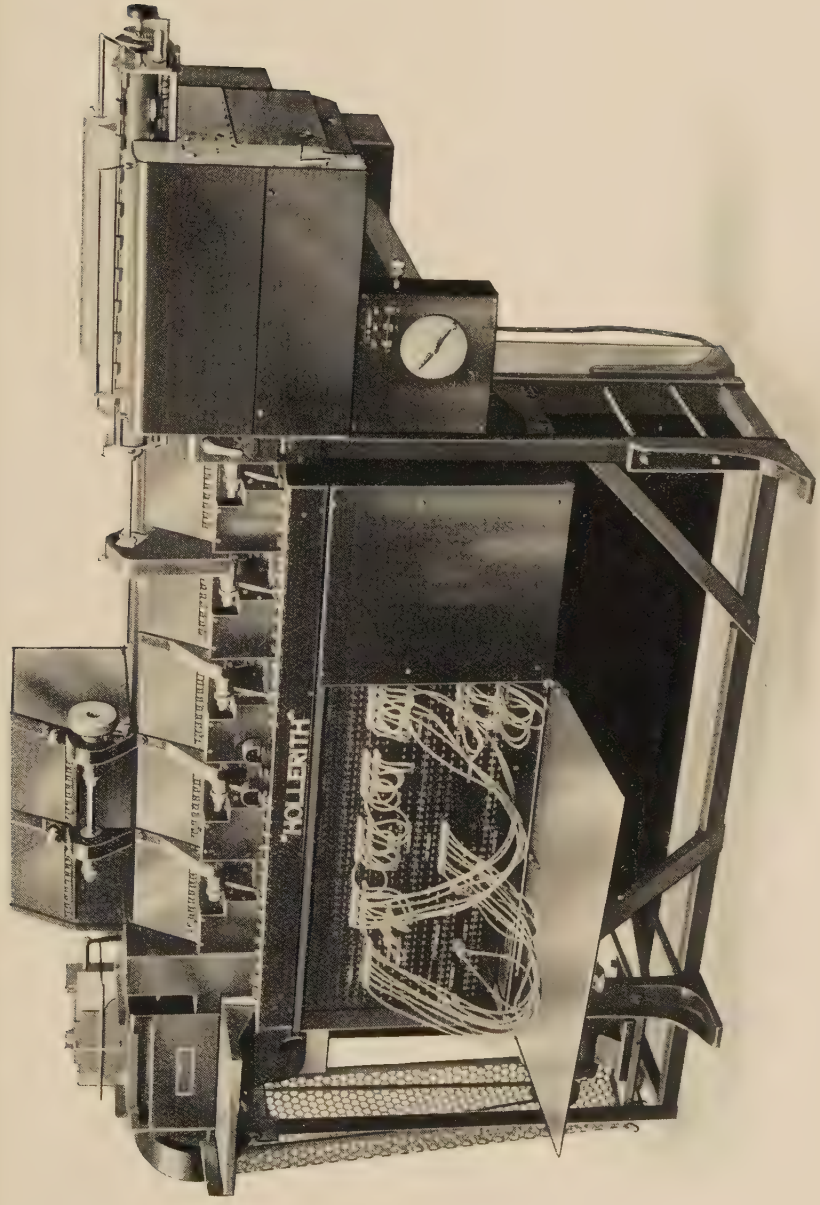


FIG. 6.—*Tabulator.*

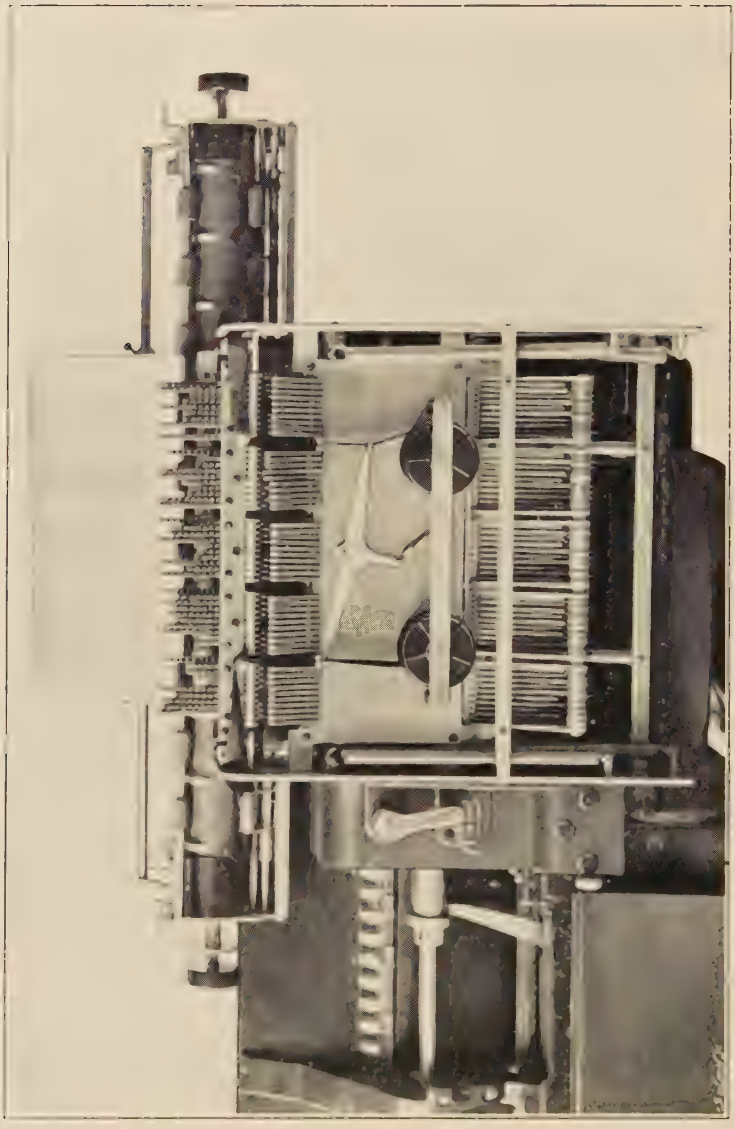


FIG. 7.—*Printing Unit.*

stop, and the operator can enter figures by hand in the usual way. As soon as a punched column is reached the duplicating process continues.

A first-class operator will punch 300 cards an hour. The number of errors made in punching is only a fraction of the errors that would be made in copying. The method of checking the punching will be described later.

The Sorter.—The function of the sorter (fig. 5) is to separate the cards into groups according to the holes punched in some selected column. Thirteen receptacles are provided for the sorted cards, one corresponding to each of the twelve positions Y . . . 9 in which a hole may be punched, and one for cards in which no hole is punched. The machine may be used to arrange cards in numerical order. In this case they are first sorted on the units column ; the 10 receiving pockets used are then emptied in order from 0 to 9, and the cards once more sorted, this time on the tens column. The process is repeated until the cards have been sorted on every column in which the digits concerned occur. The accuracy of each sort is easily verified by passing a knitting needle through the appropriate holes in all the cards withdrawn from any one pocket, or by holding the cards up to a source of light. The sorter works at the rate of 24,000 cards an hour.

The Tabulator.—The purpose of the tabulator (fig. 6) is to add the items on the cards and print the totals. The process of printing each individual item as well as the totals is known as listing ; if totals only are printed the process is tabulating. The cards are fed past a series of 45 brushes, to which direct current is supplied. Behind the cards are 45 insulated blocks, each connected to a plug-socket in a plugboard. The adding mechanism consists of a bank of five counters, each with nine separate adding wheels. Each wheel is represented on the plugboard by a plug-socket, so that any card column may be connected to any adding wheel by plugging across from one hole to another by means of a flexible cord, as on a telephone switchboard. The card acts as an insulator between brush and block ; wherever a hole has been punched current passes through, and actuates the adding wheel that has been plugged to the column containing the hole.

The printing mechanism (fig. 7) is on the right of the machine. The type-bars are lifted in phase with the passage of the card past the reading brushes ; the current that causes the adding also releases a pawl that arrests the type-bar, holding it at its correct level until the type is struck by the type-hammer. The printing of any individual column may be suppressed by locking its hammer. When totals are printed any counter may be cleared, or it may print only a "progressive total" or "sub-total," retaining its contents intact.

The speed of the machine when listing is 4500 cards an hour, and 9000 when tabulating. A delay of one second occurs each time a total is printed, so that the effective speed depends on the average number of cards per group. There are, of course, operating stops for various causes, *e.g.* placing new cards in the feeding mechanism, or new paper in the printing mechanism, so that the above speeds are not attained in practice.

Automatic Control.—This feature is of the utmost importance in the present application. When the cards corresponding to any one date have been added, the feeding must cease while the total is being printed, the

counters must be cleared and then the feeding must be resumed. This sequence is performed automatically, without any attention whatsoever on the part of the operator. Actually the cards are read by two rows of 45 brushes, the lower brushes being responsible for actuating the adding wheels as already described. The upper brushes at the same time read the following card. The control is plugged so that an effort is made to pass current through corresponding holes at the upper and lower brushes, and then through relays. In fig. 8 the machine is controlling on a 3-column field, both the cards shown being punched 426. If by the time the 0's have

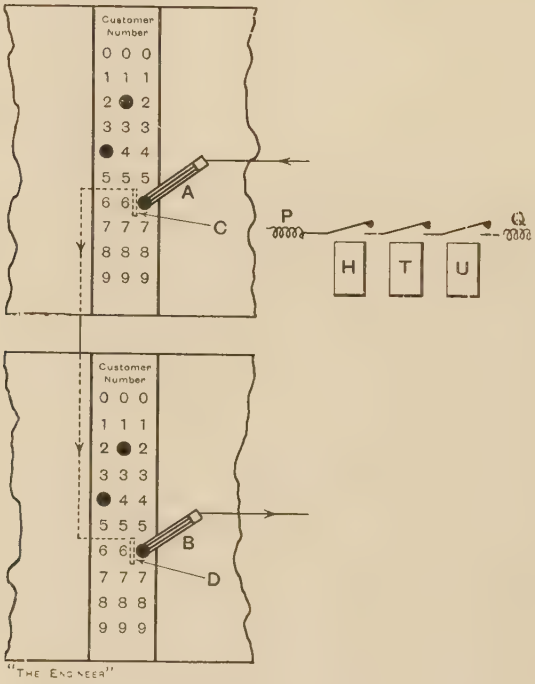


FIG. 8.—*Principle of Control.*

passed the brushes the circuit PQ is closed, the machine will continue to feed cards; if not, it will stop feeding, print the total, clear, and then continue feeding. In this case, when the line of 6's reaches the brushes the relay U is closed, when the 4's pass H is closed, and when the 2's pass T is closed, so that the upper card would be passed on to the lower brushes for addition. If the following card were punched 427, the relay U would not be closed, so that the end of the group of numbers 426 would be detected.

Verification of the Punching.—In commercial practice punching is usually verified by a special verifying punch. In order to place the responsibility for the accuracy of the cards, not on junior punch operators, but on experienced proof-readers, they were listed by means of the tabulator, and these printed lists were then compared with Brown's *Tables*.

Calculation of Long-period Terms of Nutation.—The preparation of a table for the term $-1''{\cdot}272 \sin 2L$ has been described. Each entry of the

table is now punched on a card, as shown in fig. 2, the punching of card No. 5 being illustrated. Since the contribution of any term may be positive or negative, two fields are provided ; if the contribution is positive it appears in the right-hand field, with its complement in the left-hand field, and *vice versa*. Since the nutation in obliquity depends on the cosines of the arguments on which the nutation in longitude depends, it is included on the same cards, the term in this case being $+0''\cdot551 \cos 2L$.

It is now necessary to determine the starting-point in the table, or rather in the stack of cards. The value of $2L$ is

1936 January $1^d\cdot0$	$198^\circ\cdot96142 = 1015^c\cdot452$
1950 January $1^d\cdot0$	$200^\circ\cdot16244 = 833^c\cdot326$

We have already seen that the card increment per day is $1^c\cdot000171$. Hence $365^d = 365^c\cdot062$, $366^d = 366^c\cdot063$, and 4 years or $1461^d = 1461^c\cdot250 = 0^c\cdot250$. Commencing on 1936 January 1 the card used is 1015, so that, using the cards in sequence, the card to be used on 1937 January 1 is $1015 + 366 = 1381$. Including the fraction, the card required is 1381·515, and, as discontinuities are introduced (when required, as in this case) at the beginning of a year, card 1380 is used on 1936 December 31, and $1380 + 1 + 548 - 1461 = 468$ on 1937 January 1. Thus a table of cards for January 1 of each year is prepared.

1936	1015	1940	102	1944	102	1948	102
1937	468	1941	468	1945	468	1949	468
1938	833	1942	833	1946	833	1950	833
1939	1198	1943	1198	1947	1198		

The next discontinuity is made at the beginning of the year 1953.

We may now conceive the cards for each harmonic to be arranged in a stack, with the card required for the initial date on top. The problem is to add together for the first date the entries on the top cards of each stack, then for the next date the numbers on the next cards, and so on. The top cards are picked up by hand, and the groups thus formed constitute another stack, which is fed into the tabulator. The printed result for a portion of 1938 January is shown in fig. 9. The first counter is connected to column 2 and (omitting two wheels) to columns 3–6. The second counter records the date, punched in columns 25–27 of the "main-term" card (depending on the Moon's node, and with a slowly changing coefficient), which is not permanent, but is punched specially for each date. The third and fourth counters contain the nutation in longitude and obliquity respectively. In the year 1938 the nutation in longitude is always positive and that in obliquity always negative, so that columns 12–16 and 17–20 are plugged. In years where either nutation changes sign (they cannot both change sign in the same year) the fifth counter is also used, and both fields of the nutation which passes through zero are plugged. By means of the hammer-blocks only five figures are allowed to print in the longitude totals and four in the obliquity totals, so that carry-over from the complements used does not appear. Thus the nutation, as well as its complement, will be printed ; it is easy to distinguish

these. The nutation is positive when the right-hand column contains the true value, and *vice versa.*

Reverting to fig. 9, it is seen that on the first day of the month the cards are listed, whereas on the other days only the totals are printed. This gives a printed record of the card numbers in use on that date. It will be seen that card No. 833 has been used for argument 2 ($=2L$), in accordance with the table above. The cards for subsequent days of the month follow in numerical order except those for argument 3, in which they follow in descending order.* There are eight numbered cards in each group, so that the total

Arg. No	Card No	Month	Day	Longitude	Obliquity
1	0000			15524	4007
2	0833			437	517
3	1213			164	9944
4	1460			99996	
5	0068			16	21
6	0571			99992	9992
7	0157			99992	5
8	1005			99995	
9	0023			2	
45	5330	1	01	16118	4486
	+6				
45	5336	1	02	16157	4488
	−1455				
45	3881	1	03	16194	4487
	+6				
45	3887	1	04	16233	4488
	+6				
45	3893	1	05	16269	4487
	+6				
45	3899	1	06	16305	4483
	+6				
45	3905	1	07	16339	4482
	+6				
45	3911	1	08	16374	4478
	+6				
45	3917	1	09	16407	4474
	+6				
45	3923	1	10	16439	4470

FIG. 9.—*Long-period Nutation.*

of the card numbers, which is recorded, should progress $+7-1=+6$ from day to day, unless the cards of one argument reach their highest number, to be followed by card No. 1. The "period" of argument 4 ($=L-\pi$) is also 1461 cards (although it cannot be combined with argument 2), and since 1460 is used on January 1 and 1461 on January 2, card No. 1 will be used on January 3, making the difference between the totals for January 2 and January 3 $+6-1461=-1455$.

It will be seen that the numbers in the first counter provide all the necessary proof that the correct cards have been used. The constant total 45 shows that nine cards have been used, one from each of the arguments 1–9; the list on the first day of the month shows what cards were in use then, and this list can be compared with that prepared beforehand; the progression

* This argument is 2Ω. The cards were inadvertently prepared on the assumption that 2Ω increased, but the error was easily rectified by using the cards in the reverse order.

of the totals shows that the cards subsequently used were in a continuous numerical sequence.

The "control," which enables the tabulator to distinguish the end of one group and the beginning of the next, is on column 2, in which the nine cards of each group are punched

		o	o	o	o	o	o	o	o
1	1	1	3	4	5	6	7	8	9
		2							

respectively. Towards the end of the group control is being maintained through the o's, but when the last card punched o and 9 is followed by a card punched 1, the control is "split" and the total printed. In the next group the first two cards are connected by their common 1's, and thereafter control is maintained by the o's. In adding or sorting on a multi-punched column only the highest digit punched is effective.

The cards are finally passed through the sorter, sorting on column 2, so that they are separated into nine groups, each group containing the cards belonging to one argument. These groups are then placed *below* the unused numbers of their stack, in order to maintain the sequence. This replenishes the supply of cards, so that the mixing process, tabulation, sorting and replacement proceed continuously until the entire calculation is completed. Thus each card is used over and over again, although with different partners on each occasion. The tabulation for one year takes less than an hour.

Single-entry Tables.—All of Brown's single-entry tables were added in the manner described for long-period nutation. No complements are required, as all the quantities in the *Tables* have been rendered positive by the addition of constants, the sum of these constants being disposed of by various means, for example by subtraction from the mean longitude. The proportional parts are not negligible, so they are computed separately by hand, and applied at some convenient stage. Thus the Hollerith handles only the tabular entries, and does not relieve the computer of his interpolation. Actually no reference is made to a table when computing its proportional parts, as these are computed as cosine curves with the same period as the table, but differing by 90° in phase and having a small varying amplitude.

Double-entry Tables.—There are 61 tables each depending on two arguments, one of which is D, the mean elongation of the Moon from the Sun, while the other receives one of the numbers 1–22. These fall into four groups—one for longitude, two for latitude and one for parallax. Hence all the tables depending on any one of the arguments 1–22 are punched on one card (see fig. 2) and are added simultaneously, using four of the adding counters, the remaining counter being used for printing arguments and dates. Here the tabular entries corresponding to each argument are not numbered consecutively, but are divided into "columns," each of which contains 63 entries, one each for every half-day of the synodic period, together with an overlap of two dates at each end of the lunar month. The cards are, of course, punched with argument and column number, together with the day of the lunar month. For each lunation one "column" must be selected from each

argument, and corresponding dates added. But since all the cards that have to be added on any one date have a common number, namely, the day of the month, they are brought together, not by hand, but much more expeditiously by the sorter. In tabulation this number becomes the control number. As before, the first date of each month is listed (fig. 10), giving a permanent record of the "column" numbers used. Here the check consists of the constancy of the printed total in the first column. After tabulation the cards are sorted on the columns containing the argument number, and replaced in storage until required again.

Table and Column	Day of Month	III	IV a	IV b	V
1	056	995	3818	975	256	1395
2	026	000	1406	463	60	2302
3	073	000	69	208	184	2056
4	116	000	3799	462	32	605
5	017	000	1398	233	9	341
6	075	000	1188	117	15	143
7	003	000	573	102	14	40
8	035	000	129	21	12	6
9	038	000	140	31	9	30
10	040	000	187	14		
11	031	000	97	7		
12	006	000	39			
13	032	000	89			
14	015	000	74			
15	026	000	25			
20589		995	13031	2633	591	6918
20589		000	13324	2721	587	6967
20589		005	13492	2807	591	6978
20589		010	13532	2883	600	6952
20589		015	13455	2941	618	6884
20589		020	13277	2980	644	6785
20589		025	13023	2993	677	6651
20589		030	12720	2986	705	6492
20589		035	12395	2955	737	6314
20589		040	12081	2906	765	6130
20589		045	11807	2846	789	5946
20589		050	11585	2778	805	5775

FIG. 10.—*Brown's Double-entry Tables.*

It was in this particular section of the work that the greatest efficiency was attained; on the average 20 figures were added per second, and the totals printed. The totals from 44 tables for two years (1460 dates) were done by three girls and the machines in one day.

Calculation of Short-period Terms of Nutation.—There are four short-period quantities that depend on the same arguments, namely, the nutation in longitude ($\Delta'\psi$), the independent day-number f', and the two Besselian day-numbers A' and B'. Although f' and A' are constant multiples of $\Delta'\psi$, it was found convenient to compute all four of these quantities independently and simultaneously. The largest coefficient, in the units adopted, which are one-tenth of those in the *Nautical Almanac*, is 405, and the sum of the coefficients is less than 1000 units. The cards are punched in direct and complementary form (fig. 2) as in the case of the long-period nutation. Eleven terms have been included, and a twelfth card (punched in columns 11–14)

is necessary in each group in order to be able to list the date. Hence in column 17 (the sorting column) all the available 12 positions are used.

In the specimen for 1948 January, shown in fig. 11, the left-hand group of the first counter is the sum of columns 17–21, and advances 11 each date, except when one of the arguments recommences its cycle. The right-hand group represents the date. The second counter is devoted to $\Delta'\psi$, the left-hand group being the sum of columns 22–24, and the right-hand group the sum of columns 25–27. Figures to the left of these totals, which would result from the carry-over produced by complementary numbers, are suppressed by locking the appropriate hammers. It will be seen that two

Card Numbers	Date	Δ'ψ		f'		A'		B'	
	101	−	+	−	+	−	+	−	+
559	000	833	167	898	102	668	332	50	950
10058	000	969	031	981	019	939	061		
20406	000	967	033	980	020	935	065	995	005
30877	000	987	013	992	008	974	026	10	990
40199	000	985	015	991	009	970	030		
50293	000	11	989	7	993	23	977	999	001
60404	000	994	006	996	004	988	012		
70062	000	994	006	996	004	988	012	999	001
80246	000	3	997	2	998	5	995	3	997
90728	000	2	998	1	999	4	996	2	998
154	000	996	004	998	002	992	008		
53986	101	741	259	842	158	486	514	58	942
+11									
53997	102	815	185	888	112	636	364	94	906
−236									
53761	103	919	081	950	050	840	160	109	891
+11									
53772	104	22	978	13	987	47	953	96	904
−398									
53374	105	101	899	62	938	204	796	66	934
+11									
53385	106	141	859	86	914	280	720	21	979
−287									
53098	107	136	864	84	916	269	731	978	022
+11									
53109	108	93	907	57	943	189	811	941	059
+11									
53120	109	28	972	17	983	53	947	919	081
+11									
53131	110	951	049	972	028	905	095	910	090

FIG. 11.—*Short-period Nutation.*

complementary totals are given for each date. If the right-hand total is smaller than that on the left it is to be taken, with a + sign prefixed; if the left-hand total is smaller it is taken, with a − sign prefixed. In actual practice this leads to taking about seven successive totals from one column, followed by about seven successive totals from the other column. Thus the results in the example are read, after rounding off—

Jan.	$\Delta'\psi$	f'	A'	B'
	″	s		″
1	+·26	+·016	+·0051	−·06
2	+·18	+·011	+·0036	−·09
3	+·08	+·005	+·0016	−·11
4	−·02	−·001	−·0005	−·10
5	−·10	−·006	−·0020	−·07
6	−·14	−·009	−·0028	−·02
7	−·14	−·008	−·0027	+·02
8	−·09	−·006	−·0019	+·06
9	−·03	−·002	−·0005	+·08
10	+·05	+·003	+·0010	+·09

After the cards are mixed the time required to produce the results for one year is less than an hour, or about one minute for the calculation of each of the four elements for one month.

Work done with the Machines.—A Hollerith installation was used in H.M. Nautical Almanac Office for seven months in 1929; actually punching was started six months before the arrival of the sorter and tabulator, as it was necessary to punch 20,000,000 holes in half a million cards. The work described on long- and short-period nutation, and on double-entry tables, as well as that of most of the single-entry tables, was carried to the year 2000. The greater part of the cost was incurred in doing the first ten years, which would have sufficed for immediate needs. But to continue for the next 55 years with a trained and organised staff added very little to the cost, and was certainly more economical than re-training and re-organising ten years later. Moreover, there is little likelihood of Brown's *Tables* being superseded before the end of the century; any acquisition to our knowledge of the Moon during the next seven decades is almost certain to be expressed in the form of corrections to Brown's *Tables*, not in the form of new tables.

The sorter and tabulator, which may only be hired, cost about £2 a day. The cost of doing by hand what was done on the machines has been estimated at £6000; this estimate probably errs on the low side. The cost with the machines was certainly less than £1500.

Other Applications of the Machines.—The use of Hollerith machines in connection with systematic interpolation to tenths has already been described.* The tabulator may be used to produce values of $y = a + bx$ where x varies uniformly. A series of cards, all of which are punched 1 in column (say) 21, 2 in column 22, . . . 9 in column 29, is used. a is first entered into the adding counter. If b is, for example, 40337, column 27 is plugged to the units wheel, column 23 to the tens and hundreds wheels, and column 24 to the fifth wheel. Thus each card adds 40337 to a, and a progressive total may be printed automatically after each addition. If b is large, several counters may be connected together. Thus on one occasion the first thousand multiples of each of four 20-figure numbers were produced in four hours, *i.e.* in about one-tenth of the time in which they could have been copied by hand.

The machine may be used very efficiently for producing sums of products if the number of products entering into each sum is large. This is applicable in multiple correlation, where characteristics a, b, c . . . are observed, and $[a^2]$, $[ab]$, $[ac]$ . . . $[b^2]$, $[bc]$, etc. are required. Suppose a, b, c . . . to be punched. Sort the cards on the units column of a, and add the various columns for each digit separately, getting totals a_9, a_8 . . . a_1, b_9 . . . b_1, c_9 . . . c_1, etc. Now sort on the tens column of a and repeat the adding process, obtaining a_9' . . . a_1', b_9' . . . b_1', etc. If, for simplicity, a is less than 100, we have

$$[a^2] = 9a_9 + 8a_8 + \ldots a_1 + 90a_9' + \ldots 10a_1'$$
$$[ab] = 9b_9 + 8b_8 + \ldots b_1 + 90b_9' + \ldots 10b_1'$$

* *M.N.*, 88, 518, 1928.

and so on. Actually, instead of printing the totals $a_9 \ldots a_1$, etc., progressive totals are printed, *i.e.*

$$a_9,\ a_9+a_8,\ a_9+a_8+a_7 \ldots a_9+a_8+a_7+a_6+a_5+a_4+a_3+a_2+a_1.$$

The sum of these is $9a_9+8a_8+ \ldots a_1$. In this way, on one occasion 15,000 products of 3-figure numbers were formed and added in about three hours.

The principal method used in tide prediction is the summation of harmonic terms, so that the use of the machine in this connection suggests itself. It would yield the height of the tide at uniform intervals of time, say one hour. As long as tidal predictions are limited, as at present, to the times and heights of high and low water, the method would not be economical. But in busy ports it must surely be an advantage to know the actual height of the tide at any moment, which could be obtained from a table of hourly heights. When the demand for this is felt, the use of the Hollerith may be considered. The apparent difficulty of the varying amplitudes of the constituents for different ports would be easily overcome by using two similar sets of cards for each harmonic; by adjusting the relative phases of these, any amplitude from zero to twice that of either set may be obtained.

THE FORMATION OF SPIRAL ARMS.

B. M. Peek, M.A. (Plate 14.)

In this paper an attempt is made to investigate the motion and configuration of particles of matter, thrown off from a nebulous mass during a period when the mass is rotating and contracting.

In the early stages of the argument we follow in detail the theory of the evolution of the great nebulæ put forward by Sir James Jeans.*

Jeans deduces that, in a primeval nebulous mass, owing to gravitational instability, condensations will form, and that the nebular material will be drawn in toward these nuclei by gravitational attraction. At first gravity alone will control the motion of the falling particles, but as soon as the masses have become sufficiently condensed, gas pressure will begin to retard the motion. Any translational currents in the primeval nebula will have the effect of giving angular momentum to the condensing mass, and if the angular velocity reaches the value given by $\frac{\omega^2}{2\pi\gamma\bar{\rho}}=0{\cdot}3$ approximately, where ω is the angular velocity, γ the gravitational constant, and $\bar{\rho}$ the average density of the mass, the nebula will assume a lenticular form and will proceed to shed matter round its sharp edge. As the nebula continues to contract, ω increases according to the law $\omega^2r^3=\gamma M$, where r is the radius and M the mass of the nebula, since the particles at the rim are moving in circular orbits under gravitation alone. Such a rotating mass would shed matter indiscriminately all round its sharp edge, but for the fact that the resultant tidal

* *Astronomy and Cosmogony.*

forces, however small, due to the rest of the matter in space, determine two preferential points on the circumference, 180° apart, at which the ejection will begin. The use of the words "ejection" and "ejected" must not be taken to imply anything in the nature of an expulsion of the matter; particles are simply left behind, rotating in circular orbits under gravity, as the mass contracts.

From this point the present investigation proceeds on the following four assumptions :—

(1) That the density of the ejected matter is so small that the effect of the whole in perturbing the orbits of its individual particles, moving under the gravitational influence of the central mass, is negligible.

(2) That the mass of the central nebula remains sensibly constant. This is consistent with assumption (1).

(3) That there is a law connecting the rate of contraction of the nebula with its radius. This law will have the form $-\lambda\frac{dr}{dt}=\phi(r)$, where λ is a constant.

(4) That as soon as appreciable ejection has taken place at the two antipodal points, the effect of this ejected matter in raising tides will, owing to its proximity, completely outweigh the original cosmic tidal forces, so that the points of ejection will rotate with the matter that has just left the nebula, and will always lie immediately below the last particles to have been ejected. This assumption is the crux of the present paper, and is at variance with what has usually been assumed, namely, that the tidal forces from without cause the points of ejection to remain fixed in space. It is not, as may appear at first sight, inconsistent with assumption (1), for a particle at some distance from the nebula will be under the perturbing influence of particles on either side of it, whose effects will cancel out to the order of approximation here considered, whereas particles at the rim are in equilibrium, except for the force exerted by the contiguous ejected matter, which acts in one direction only.

It is easy to calculate the positions of the two points of ejection at any time after the ejection first commenced, for, since the rim of the nebula is in rotation in a circular orbit about the central mass, the instantaneous angular velocity of the matter which has just been shed is the same as that of the rim, and the two points of ejection will therefore rotate with the nebula as though they were objects floating upon its surface; these positions are given by equations (4) below.

In considering possible forms of the function $\phi(r)$, which gives the law for the rate of contraction of the nebula, we assume that ejection of matter would not have commenced until after the influence of the gradually increasing pressure in retarding the contraction had begun to outweigh the inward acceleration due to gravity. In these circumstances $\frac{d^2r}{dt^2}$ must be positive, and $\frac{d\phi(r)}{dt}$ therefore negative. Accordingly, $\phi(r)$ decreases with the

time, that is, with decreasing r. The number of possible forms of such a function being, of course, unlimited, we confine ourselves here to the treatment of a few comparatively simple cases, and examine the distributions of the ejected matter to which they lead us. The general similarity of the results tends to show that precise definition of this law is not of great importance.

Simple forms which suggest themselves are :

(a) $\phi(r) = r^n$.
(b) $\phi(r) = r$. {Special case of (a)}
(c) $\phi(r) = \log r$.
(d) $\phi(r) = a^r$.

It can be seen at once that (d) is not a reasonable form, for $\frac{dr}{dt}$ tends to $-\frac{1}{\lambda}$ as r approaches zero, and the nebula will shrink to a point in a finite time.

Cases (a), (b) and (c) are now investigated.

r_0 is the radius of the nebula when ejection of matter first commenced, and r_p its present radius.

T_q is the time when the radius of the nebula was r_q, T without suffix corresponding to arbitrary radius r.

ψ_q is the angle through which the nebula has rotated during contraction from radius r_0 to radius r_q, *e.g.* $\psi_0 = 0$; ψ without suffix is the corresponding angle for a contraction from radius r_0 to arbitrary radius r.

Case (a) : Rate of contraction proportional to some power of the radius.

$$-\lambda\frac{dr}{dt} = r^n. \qquad (1)a$$

Hence

$$T - T_0 = \frac{\lambda}{1-n}(r_0^{1-n} - r^{1-n}).$$

Clearly, in the problem we are considering n must not be less than unity, otherwise r becomes zero in the finite time $t = \frac{\lambda r_0^{1-n}}{1-n}$.

The equation for $T - T_0$ therefore becomes

$$T - T_0 = \frac{\lambda}{n-1}\left(\frac{1}{r^{n-1}} - \frac{1}{r_0^{n-1}}\right). \qquad (2)a$$

The case of $n = 1$ is dealt with separately under case (b).

Again, the angular velocity of the rim of the nebula, and therefore of the points of ejection, at any time is given by $\omega^2 r^3 = \gamma M$.

Hence, putting $\gamma M = \kappa^2$, we have

$$\frac{d\psi}{dt} = \kappa r^{-\frac{3}{2}}, \qquad (3)a$$

and on combining equations (1) and (3),

$$\frac{d\psi}{dr} = -\kappa\lambda r^{-(n+\frac{3}{2})}.$$

The angle swept out by these points, during contraction of the nebula from radius r_0 to radius r, is therefore given by

$$\psi - \psi_0 = \int_{r_0}^{r} -\kappa\lambda r^{-(n+\frac{3}{2})} dr$$

$$= \frac{\kappa\lambda}{n+\frac{1}{2}}\left(\frac{1}{r^{n+\frac{1}{2}}} - \frac{1}{r_0^{n+\frac{1}{2}}}\right). \qquad (4)a$$

Now if a nebula started to shed matter from its rim when its radius was r_0, and has since contracted to its present radius r_p, a particle of ejected matter, now observed to lie at a distance r from the centre, will have spent part of the time, $T - T_0$, rotating with the central mass, during which the points of ejection will have turned through an angle α given by equation (4), and the rest of the time, $T_p - T$, revolving freely in a circular orbit with angular velocity $\kappa r^{-\frac{3}{2}}$, during which its radius vector will have swept out an angle β, where

$$\beta = \kappa r^{-\frac{3}{2}}(T_p - T).$$

Since $T_p - T$ is given by equation (2), the present configuration of all the ejected matter must be in accordance with the equation

$$\theta = \alpha + \beta = \frac{\kappa\lambda}{n+\frac{1}{2}}\left(\frac{1}{r^{n+\frac{1}{2}}} - \frac{1}{r_0^{n+\frac{1}{2}}}\right) + \frac{\kappa\lambda r^{-\frac{3}{2}}}{n-1}\left(\frac{1}{r_p^{n-1}} - \frac{1}{r^{n-1}}\right)$$

$$= \frac{\kappa\lambda}{(n+\frac{1}{2})(n-1)}\left\{(n+\tfrac{1}{2})r_p^{-(n-1)}r^{-\frac{3}{2}} - \tfrac{3}{2}r^{-(n+\frac{1}{2})} - (n-1)r_0^{-(n+\frac{1}{2})}\right\}. \qquad (5)a$$

Figs. 1 and 2 show this distribution plotted for the values $n = 2$ and $n = \frac{3}{2}$, for which equation $(5)a$ becomes

$$\theta = \kappa\lambda\{r_p^{-1}r^{-\frac{3}{2}} - \tfrac{3}{5}r^{-\frac{5}{2}} - \tfrac{2}{5}r_0^{-\frac{5}{2}}\} \qquad (5)a\ (n=2)$$

and

$$\theta = \kappa\lambda\{2r_p^{-\frac{1}{2}}r^{-\frac{3}{2}} - \tfrac{3}{2}r^{-2} - \tfrac{1}{2}r_0^{-2}\} \qquad (5)a\ (n=\tfrac{3}{2})$$

respectively.

The argument for cases (*b*) and (*c*) is precisely the same as for (*a*). For these, therefore, the equations only are given, similar numbers to the equations denoting similar stages in the analysis.

Case (*b*).

We have

$$-\lambda\frac{dr}{dt} = r, \qquad (1)b$$

whence

$$T - T_0 = \lambda \log \frac{r_0}{r}. \qquad (2)b$$

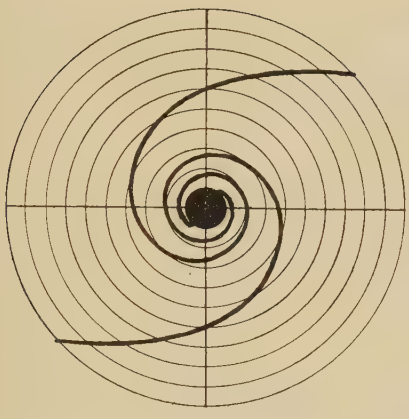


Fig. 1.

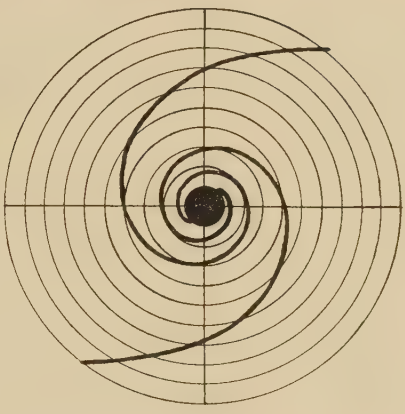


Fig. 2.

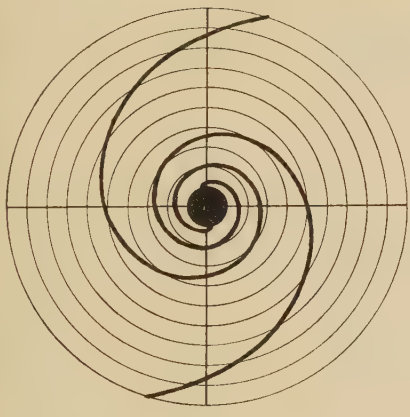


Fig. 3.

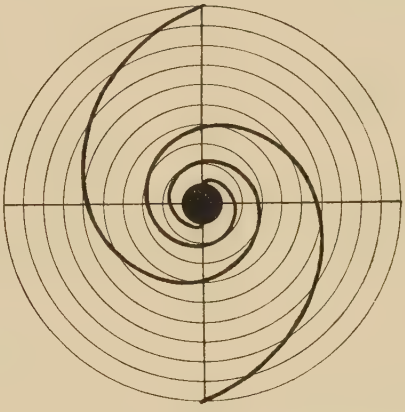


Fig. 4.

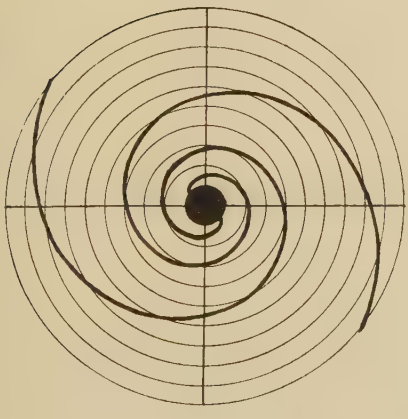


Fig. 5.

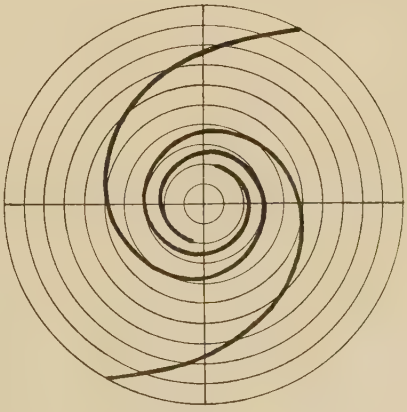


Fig. 6.

Again,
$$\frac{d\psi}{dt} = \kappa r^{-\frac{3}{2}}, \tag{3}b$$
whence
$$\psi - \psi_0 = \tfrac{2}{3}\kappa\lambda(r^{-\frac{3}{2}} - r_0^{-\frac{3}{2}}). \tag{4}b$$
Lastly,
$$\theta = \tfrac{2}{3}\kappa\lambda(r^{-\frac{3}{2}} - r_0^{-\frac{3}{2}}) + \kappa\lambda r^{-\frac{3}{2}} \log \frac{r}{r_p}$$
$$= \kappa\lambda\left\{\tfrac{2}{3}(r^{-\frac{3}{2}} - r_0^{-\frac{3}{2}}) + r^{-\frac{3}{2}} \log \frac{r}{r_p}\right\}. \tag{5}b$$

This distribution is shown plotted in fig. 3.

Case (*c*) : Rate of contraction proportional to the logarithm of the radius.
$$-\lambda\frac{dr}{dt} = \log r, \tag{1}c$$
whence
$$T - T_0 = \int_{r_0}^{r} \frac{-\lambda dr}{\log r}. \tag{2}c$$
Again,
$$\frac{d\psi}{dt} = \kappa r^{-\frac{3}{2}}, \tag{3}c$$
whence
$$\psi - \psi_0 = \int_{r_0}^{r} \frac{-\kappa\lambda r^{-\frac{3}{2}} dr}{\log r}$$
$$= \kappa\lambda \int_{r^{-\frac{1}{2}}}^{r_0^{-\frac{1}{2}}} \frac{dz}{\log z} \tag{4}c$$
on putting $z = r^{-\frac{1}{2}}$.

Lastly,
$$\theta = \kappa\lambda \int_{r^{-\frac{1}{2}}}^{r_0^{-\frac{1}{2}}} \frac{dz}{\log z} + \kappa\lambda r^{-\frac{3}{2}} \int_{r_p}^{r} \frac{dr}{\log r}$$
$$= \kappa\lambda[li(r_0^{-\frac{1}{2}}) - li(r^{-\frac{1}{2}}) + r^{-\frac{3}{2}}\{li(r) - li(r_p)\}], \tag{5}c$$
where $li(a)$ stands for $\int_0^a \frac{dx}{\log x}$.

Two distributions given by the last equation are shown plotted in figs. 4 and 5, for in this case the form of the curve varies somewhat according to the value chosen for r_p, as explained below.

Equations (5) have some interesting points in common.

Obviously they all represent spirals.

The constants κ, λ appear only together as a product, which is a factor of the right-hand side. Hence the mass $M\left(=\frac{\kappa^2}{\gamma}\right)$ of the nebula, and the value chosen for λ, simply combine in determining how many convolutions of the

spiral will correspond to a given contraction of the nebula, and do not affect its general form.

The spirals resemble only over part of their course the equiangular spirals, with which the arms of actual nebulæ have often been compared. If we differentiate the equations, we have in each case $r\frac{d\theta}{dr}=0$, for $r=r_p$. The spiral arms, therefore, always issue from the nucleus in directions making a right angle with its rim. This fact may, of course, be directly inferred from the consideration that the angular velocity of the matter immediately contiguous to the rim is the same as that of the rim itself. A second differentiation shows that $r\frac{d\theta}{dr}$ changes very rapidly as r increases above r_p, reaches a stationary value, and then slowly tends to zero again as r approaches infinity. The spirals, therefore, all approach an asymptote, $\theta=$ constant $(=0$, for $r_0=\infty)$.

The values of r for which $r\frac{d\theta}{dr}$ is stationary, and near which the spirals are therefore equiangular, are for the different cases :

$a(n=2)$	$r=1{\cdot}67r_p$,
$a(n=\frac{3}{2})$	$r=1{\cdot}77r_p$,
b	$r=1{\cdot}95r_p$.

In case (c) it will be noted that there is a critical value, represented by $r=1$ in the equations, below which the radius cannot decrease ; for when $r=1$, $\frac{dr}{dt}=0$, and contraction ceases. In plotting fig. 4, r_p was taken to be twice the critical radius, for fig. 5 one hundred times. The corresponding values for which $r\frac{d\theta}{dr}$ is stationary are at about $r=2{\cdot}0r_p$ and $r=2{\cdot}6r_p$ respectively.

In making the diagrams 1 to 5, the constants κ, λ have been chosen so that for a contraction corresponding to $r_0=10r_p$ there will be approximately one and a half convolutions of the spiral arms, which is very suitable for comparison with the average spiral nebula. The strong resemblance between all the curves is not surprising. The more rapid orbital revolution of the inner particles is alone sufficient to ensure the spiral form, whatever the law of contraction. As such a law only provides something in the nature of a handicap on the more swiftly moving particles, it is not to be expected that its exact form will be a dominating factor. The fastest runners have been allotted the inside of the track ; give them sufficient time and they are bound to take the lead.

A comparison of the forms of the spirals, which have been derived, with those actually observed in the sky shows that the agreement is in many instances far closer than is the case with the equiangular form. In the majority of the nebulæ the angle of the spiral increases with the distance from the nucleus, though there are exceptions.

The plate shows, superimposed upon a photograph of the nebula M81, a projection of the curve given in fig. 6. The latter is part of a law (*a*) spiral, drawn for $n = 3$, and if completed would show about three convolutions. The fit could actually have been somewhat improved for the inner regions by taking a larger value of n, without appreciably affecting the rather striking accordance exhibited by the outer portions of the arms.

Now although the agreement between the curves that have been derived from equations (5) and the observed configurations of the actual spiral nebulæ may appear very promising, certain other quantitative conditions require to be fulfilled before any theory of the manner in which the nebulæ have evolved can be considered plausible. In brief, the time required for the development of the arms, which in accordance with the above ideas would be the time taken for a contraction of the nebula on the scale observed, must be of a reasonable order of magnitude.

It must be admitted at once that with the values at present accepted for the masses and diameters of the spiral nebulæ it seems impossible to arrive at a time of contraction even of the right order of magnitude to be acceptable to any school of cosmogonists. It may be of interest, however, to examine the extent to which the current values would need to be revised before a reasonable value for the time could emerge.

To serve as an example let us take 6×10^{41} grammes as the mass of the nucleus and 5500 parsecs as the over-all diameter of a spiral nebula. The first figure is a little in excess of the average mass of $2{\cdot}6 \times 10^8$ suns given by Hubble,* which, he states, must be regarded as a lower limit. The second is probably a conservative estimate for M81 ; this nebula, however, though of only about one-third of the diameter of M31, is considered by many to be considerably more extensive than the average, so that both the figures chosen tend to make the derived time rather longer than may be strictly justifiable in the present state of our knowledge.

Adopting these values, however, and putting $\psi_p - \psi_0 = 4\pi$ and $r_0 = 12r_p$, say, we can determine the value of the constant λ from equations (4), and hence the time taken for contraction, $T_p - T_0$, from equations (2). This comes out as follows for the different cases :—

$a(n=2)$	$0{\cdot}86 \times 10^8$ years
$a(n=\frac{5}{2})$	$1{\cdot}1 \times 10^8$,,
b	$1{\cdot}4 \times 10^8$,,
$c(r_p=2)$	$1{\cdot}9 \times 10^8$,,
$c(r_p=100)$	$2{\cdot}6 \times 10^8$,,

If we choose other values for the mass and diameter, we find that the time varies exactly as $r_0^{\frac{3}{2}} M^{-\frac{1}{2}}$ for cases (*a*) and (*b*), and approximately so for case (*c*) ; in all cases it is directly proportional to the number of convolutions.

If at any time ejection of matter ceases, the arms simply go on coiling themselves up owing to the more rapid revolution of the inner particles, and it seems reasonable to suppose that after a few more convolutions the matter constituting the arms will have become so disturbed that the spiral form will

* *Mt. Wilson Cont.*, No. 324, 1926, " Extra-Galactic Nebulæ."

Spiral Nebula M81 and Projection of Fig. 6 (p. 711).

no longer remain recognisable. It is improbable, therefore, that a nebula of the mass and dimensions considered, evolving along the lines here indicated, could still show the true spiral form, if it had begun to shed the matter now observed in the arms at an earlier date than 10^9 years ago.

If we consider the possibility of the galaxy having originally evolved thus, as a single huge spiral nebula, then, taking its greatest diameter to be 60,000 parsecs, its mass 10^{44} grammes, and ten convolutions as sufficient for its spiral form to have become unrecognisably degraded, we get something between 10^9 and 10^{10} years for the duration of its evolution, down to the time when it ceased to be recognisable as a spiral.

Thus it is obvious that with a period of 10^{13} years for the lives of the stars such a theory of the evolution of the spirals is hopelessly inadequate. There remains, however, the possibility that the latest values for the rate of expansion of the universe may call for a somewhat drastic revision of the time-scale; in this case, since the last word may still remain to be said on the subject of the masses and dimensions of the nebulæ, it seems just possible that by a judicious stretching the two ends may yet be enabled to meet without rupture. For instance, the long ages that must have elapsed during the formation of the original condensations in the primeval nebula, before the evolution of arms began, may partially bridge the gap.

In conclusion we may notice that 10^8 years is the order of the light time from the most distant nebulæ yet observed; if, therefore, the time taken for the evolution of the arms were of the same order, we might expect to find statistically a decrease in the number of convolutions exhibited by spirals at greater and greater distances.

The author wishes to record his indebtedness to Mr. W. M. H. Greaves for his kind criticism and advice during the preparation of this paper.

A REMARKABLE VARIABLE STAR (V SAGITTÆ).

P. M. Ryves. (Plate 15.)

In 1902 L. Ceraski announced the variability of the small star which later received the designation V Sagittæ. Hartwig, from observations in 1903, suspected a period of 96 days, but Harvard subsequently pronounced the variation "often rapid and probably irregular," range 9·5 to 12·0 (1907). One hundred and thirty-eight observations made at Zaragoza in 1910 and 1911 confirmed the rapid character of the light-changes and revealed a periodic tendency.

V Sagittæ is not in the B.D. Its position for 1855 is R.A. $20^h\ 13^m\ 47^s$, dec. N. 20° 52′; approximate position for 1930, R.A. $20^h\ 17^m{\cdot}1$, dec. N. 20° 53′. The galactic co-ordinates are: long. 30°, lat. S. 10°.

This star has been strangely overlooked by observers and investigators. The present study is based on visual observations made by the writer on

1243 nights during the years 1910–12 and 1923–31. In order to preserve continuity during unavoidable absences from Zaragoza, some observations have been made at other localities—Madrid, 1924 ; Cambridge, 1925 ; Leyden, 1928 ; and on several occasions at Headley, England (Rev. T. E. R. Phillips's observatory).

Some trouble has been experienced with the reduction of the estimates, partly through the lack of reliable faint standards and also on account of a change in the observer's personal equation of position, which took place during the period covered by the observations and has necessitated the use of corrections with reversed signs after the change. During the critical period of the reversal (1927–28) no corrections have been applied. There is obviously some uncertainty about the individual deductions at this time, but since the comparisons are, whenever possible, arranged in such a way that position errors cancel out, the errors resulting from inadequate correction will generally be small.

The light-scale deduced from the observations has been adjusted approximately to the photometric scale of Harvard College Observatory by means of a photograph (enlargement of No. A 2758) sent me by the late Professor E. C. Pickering. On this photograph the magnitudes of several comparison stars are marked, but, as already stated, I have not much confidence in the values attached to the fainter stars, some of which appear to be sensibly overrated. One of the comparison stars is suspected of slight variability.

The observations of 1910 showed peculiar rapid oscillations within a range of about 2 magnitudes. It was obvious that, in order to bring out the true character of the light-curve, very frequent observations would be necessary, and as it was not possible to continue observing on an adequate scale, the work was abandoned in 1912 but was recommenced in 1923 and intensified three years later. During the past six years V Sagittæ has been observed on an average of 170 nights each year (least, 149 ; most, 213).

The need for frequent observations at all costs (even an inferior estimate being often valuable) necessarily entails a low average weight, since estimates must often be made under unfavourable conditions, and this makes it necessary to seek for some indication as to how much reliance can be placed on the estimates. A study of the probable errors, or divergences from a smooth curve, which with a more regular or slowly changing star would provide a measure of accuracy, in the case of V Sagittæ can tell us little, since, on account of the irregular and unexpected nature of the variations, the divergences may represent real changes in the star's light. However, there are other ways of gauging the probable magnitude of the accidental and short-term errors which principally concern us. For example, the observations may confirm one another. When a smooth curve is drawn, indicating the general trend of the light-change, the dots representing the observations naturally fall at various distances from the curve, a random distribution suggesting errors as the cause of the scattering ; but when several dots fall systematically in line, in a manner precluding a chance

origin, we may safely conclude that they indicate a real irregularity in the star's variation, however small the amplitude may be, and by analogy other less conclusively revealed irregularities may be assumed to be real. Our curve shows several examples of this kind of mutual confirmation. Thus in 1926, J.DD. 2424,688–696, five observations indicate a fall of 0·4 magnitude; and a little later, days 2424,704–708, a rise of only 0·25 magnitude is beautifully shown by five observations in series mutually confirmatory. In 1927, J.DD. 2425,032–39, the very regular distribution of the observations during the fall may be noted. Eight observations in 1929 are distributed along a pause and rise of 0·4 magnitude, and in 1931 several similar cases are evident, *e.g.* days 2426,503–508, five observations along a fall of 0·4 magnitude (slightly irregular); days 2426,530–533, four observations on a rise of 0·25 magnitude; and days 2426,617–627, ten observations showing the star stationary, falling slowly, then more rapidly. There are many other such cases where, in tracing a most probable curve, the observations are in almost perfect agreement or within less than 0·1 magnitude of such agreement.

The reality of most of the minor sinuosities in the curve is supported by the way in which similar forms recur. We have numerous cases of (1) small secondary minima, a rapid fall followed immediately by a recovery equally rapid or nearly so, the whole very similar to the larger minima; (2) curious halts on the rise but seldom on the fall; (3) complete oscillations (*e.g.* fall-recovery-rise-return to smooth curve); and (4) the remarkable recurrence of subsidiary maxima and minima at intervals which are approximately constant during a considerable period of time (see later). Lastly, in spite of the general persistence of rapid fluctuations, there occur brief intervals of quiescence, and then it is found that the observations cluster along the smooth curve in a satisfactory manner. For example, in 1926, J.DD. 2424,731–736, a pause, quite unexpected by the observer, in the middle of a rapid rise; estimates on 5 consecutive nights are within 0·02 of their mean. The same year, days 2424,837–847, a pause between rapid rise and fall; seven consecutive observations within 0·03 of a smooth curve. In 1929, days 2425,782–815, seventeen observations showing a gradual rise are all within 0·1 of a straight curve. In 1930, days 2426,212–224, the very small oscillations, possibly due to errors of observation, at beginning of rise, give substance to the larger but still small fluctuations of the preceding days, and a similar series of accordant observations denoting another such quiet interval between more agitated periods occurs in the following year, days 2426,509–522.

From the above evidence it may be concluded that the accidental errors are generally not more than one-tenth of a magnitude, though of course larger errors are sometimes present, especially when the star is faint or badly situated, and that the majority of the minor sinuosities of the curve outside these limits represent real changes in the star's light. According to these tests the observations appear to be considerably better than was expected.

Character of the Variation.—The most noteworthy feature is the rapid

fluctuation in brightness superimposed on the slow general light-change, which it sometimes masks to a greater or less extent. V Sagittæ has been observed to decrease $1^{m}.4$ in 1 day and 2 magnitudes in 3 days, and to rise $0^{m}.9$ in 1 day and $2^{m}.1$ in 5 days. The characters may be described as follows :—

1. *A slow variation of a periodic character* and range of 2 magnitudes can be disentangled from the short-term and irregular fluctuations. The period is about 18 months (530 days suits the observations best), but there are irregularities. In 1924 the usual minimum did not take place. Part of the rise to maximum is generally rapid.

2. *A secondary variation*, which may be described as a tendency for the maxima to take the form of three peaks separated by intervals of about 130 days. After the faint period the light increases to a principal maximum (first peak) of perhaps magnitude 10·5, soon falls again to median magnitude or somewhat lower, rising to a second and generally to a third maximum usually fainter than the first. The interval is on the average about 130 days, but varies considerably. In the last two cycles these secondary maxima were well defined, but in 1928, though present, they are only slightly indicated, and in 1926 there are two smaller maxima wedged in between them. The feature, however, is quite distinct.

3. *The Rapid Fluctuations.*—Apart from some very brief intervals, short-term variations, irregular or semi-periodic, are taking place all the time. During the period of general faintness (normal minimum) they are of somewhat smaller range except for occasional deep minima of short duration, which, incidentally, constitute the absolute minima of the star. The most pronounced fluctuations take place just after the light has increased to principal maximum and on the decline to the secondary minima. (In 1926 the light fell in 3 days from 11·3 to 13·0, climbed up to 10·4 in the next 20 days, falling again in 3 days to 11·9.) At other times there are rapid oscillations with periods of 10 to 15 days and large amplitudes ($1\frac{1}{2}$ magnitudes in 1910, 2 magnitudes in 1930).

These fluctuations take several forms, the most common being a subsidiary minimum often of about 10 days' duration and of various depths from the smallest recognisable to $1\frac{1}{2}$ magnitudes. A maximum of similar dimensions is less frequent, as is also a complete oscillation, *e.g.* fall-recovery-rise-return to mean curve. Somewhat similar is a continuous zig-zag variation, but more commonly there is a tendency for the light to remain constant for a few days whenever it reaches the neighbourhood of the smooth curve, thus separating the divergences into distinct maxima and minima. This tendency to pause at about the magnitude of the smooth curve may explain some of the pauses so frequently observed on the rises.

An interesting point in connection with all these short-term variations is the fact that, however violent they may be and however large the range, they generally do not alter the general trend of the long-term variation. For example, the light may be decreasing at a rate of one magnitude in 50 days when there occurs a narrow minimum lasting 10 days. If the amount of the sudden fall is one magnitude, the recovery will be not one magnitude,

but four-fifths—that is, to the magnitude which the star would have reached if the narrow minimum had not occurred. This seems to show quite clearly that the origin of the rapid fluctuations is very superficial and quite independent of the slow long-term variation of which the origin is more deep-seated. The short-lived character of the rapid fluctuations is suggestive of condensation, although the amplitude is rather large, and it is difficult to imagine how a solid or liquid cloud sufficiently dense to cut off perhaps nine-tenths of the star's light can become dissipated in a few days.

Abortive Maxima and Minima.—Among the narrow or subsidiary maxima and minima there are many cases of failure to reach the obvious goal, due to a sudden reversal in the direction of the light-change. The *amount* of change is often just about sufficient to complete the phase, but the sign is reversed.

Periodicities.—The period of 530 days requires confirmation by future work. It suits the observations of recent years and also those of 1910–11, but the agreement is not conclusively indicative of permanency. The secondary period of 130 days, if increased to $132\frac{1}{2}$ days, or one-quarter of the former, might be considered as the true period on the assumption that every fourth maximum is generally reversed, becoming a minimum. The same cause may be supposed to produce either a maximum or a minimum, some other factor deciding which it is to be. That something of this kind happens seems evident from the phenomenon of abortive maxima. In our Sun increased activity produces bright faculæ and dark spots, and it is often difficult to decide whether the Sun is brighter or fainter on that account. On the other hand, the interval between the primary maximum and the first secondary appears to be somewhat longer than that intervening between the two secondaries, which would suggest subsiding pulsations rather than a constant periodic term.

The smaller fluctuations sometimes show a marked periodicity, which is different in different seasons. In 1910 a period of 12 ± days was in evidence, in 1927 one of 17 to 18 days, and in 1928 a very constant period of 16 days was in operation.

The possibility that V Sagittæ may be a nova or closely allied to that class is suggested by its light-curve and its position on the border of the Milky Way. The following points of similarity may be mentioned: (1) A period of faint light during which small irregular variation takes place; (2) a rapid rise to a principal maximum; (3) a rather rapid decrease and subsequent recovery to one or two secondary maxima; (4) rapid oscillatory or semi-periodic fluctuations as the light declines.

Unfortunately there is no information concerning the spectrum of V Sagittæ. A request that the spectrum should be photographed at observatories in Canada and Scotland was in both cases unfavourably entertained.

Zaragoza, Spain:
1932 *February.*

OBSERVATIONS OF NOVÆ, 1931–1932.

W. H. Steavenson.

All the observations contained in this paper were made with the 20-inch Hindle reflector. It may be convenient here to repeat the list of authorities for the comparison stars used in these and previous years, as it is some time since specific reference was made to them.

Nova	Reference for Comparison Stars
Aquilæ (1918)	*H.C.O. Circular*, 208, 1918.
Aurigæ (1892)	*M.N.*, **52**, 433, 1892 (Burnham).
Coronæ (1866)	*H.C.O.* See this paper, p. 720.
Cygni (1876)	*M.N.*, **82**, 116, 1921 (Seares).
Cygni (1920)	Own estimates. See this paper, p. 721.
Geminorum (1912)	Own estimates. Chart in *M.N.*, **72**, 516, 1912.
Lacertæ (1910)	*M.N.*, **72**, 672, 1912 (Barnard).
Ophiuchi (1848)	*M.N.*, **81**, 170, 1921 (Barnard).
Persei (1901)	*M.N.*, **80**, 721, 1920 (Seares).

Nova Aquilæ (1918)

Observed on one night only, 1931 June 3. The magnitude was then estimated to be 11·0. This is about the brightness of the nova prior to its outburst. The fading seems now to have ceased, and the light is sensibly constant. The image appears stellar, and no nebulosity can be detected visually.

Nova Aurigæ (1892)

This nova is perhaps very slightly variable. It appears sometimes fainter and sometimes brighter than a star of mag. 14·8 which follows it by about 30″. The nova, which appears perfectly stellar, was observed on ten nights as under:

Date	Mag.	Date	Mag.
1931 Oct. 5	14·7	1931 Nov. 4	14·7
7	14·9	5	14·8
8	14·8	9	14·9
14	14·9	Dec. 7	14·9
18	14·9	1932 Jan. 26	14·9

T Coronæ (1866)

This object appears to be sensibly constant in light and is quite stellar in appearance. No nebulosity can be seen around or near it. It was observed on one night only, 1931 June 14, when the magnitude was estimated at 9·96. The two comparison stars commonly used for this nova are B.D. +26° 2761, of which the Harvard magnitude is 9·76, and +26° 2764,

for which I have adopted a value of 10·06 by estimation from the other star.

Nova Cygni (1876)

Observed on thirty nights. This is one of the definitely variable novæ. Occasionally the range of brightness amounts to a full magnitude, but the changes observed in 1931 were less marked, amounting at a maximum to $0^m.42$. Nothing abnormal was noted in the image of the nova, which appeared quite stellar.

Date	Mag.	Date	Mag.	Date	Mag.
1931 June 12	14·81	1931 July 28	14·81	1931 Oct. 3	14·76
14	14·71	Aug. 14	14·61	5	14·76
20	14·86	15	14·55	7	14·81
21	14·86	16	14·57	8	14·91
24	14·61	17	14·49	12	14·81
July 6	14·76	22	14·50	14	14·76
9	14·71	25	14·66	20	14·66
11	14·81	27	14·71	Nov. 1	14·66
12	14·61	29	14·66	5	14·81
27	14·86	Oct. 2	14·76	Dec. 7	14·76

Nova Cygni (1920)

Observed on ten nights. This object is now fading very slowly and has nearly reached the brightness it had before its outburst. It appears stellar in the telescope, and no nebulosity is visible around or near it. It is conveniently surrounded by four faint comparison stars, whose positions, with regard to the nova, I have measured roughly on a photograph taken with the 30-inch reflector of the Royal Observatory, Greenwich.

Star	P.A.	Dist.	Estimated Mag.
	°	″	
A	290	27	14·0
B	182	51	15·2
C	30	38	15·8
D	105	38	16·0

The positions of the nova and its comparison stars are indicated on the chart, fig. 1.

D is near the limit of visibility with the 20-inch reflector under good conditions.

Date	Mag.	Date	Mag.
1931 June 12	15·0 ±	1931 July 12	15·0
14	15·0	28	15·0
17	14·9	Aug. 16	15·1
21	15·0	17	15·2
July 6	14·9 ±	Oct. 7	15·0

Nova Geminorum (1912)

The presence of a bright star, a few seconds of arc to the north, makes this a very difficult object to observe accurately. The conditions

were not good on the three occasions on which it was observed, and the changes indicated by the figures may not be real. The nova is now of about the same brightness as before its outburst. As in former years, I had repeated impressions of a faint symmetrical nebulosity immediately surrounding it, but the image itself is stellar. The two comparison stars used will be found on the chart in *M.N.*, **72**, 516, 1912. It should be noted that the nova is the more southerly of the two objects within the circle, which is

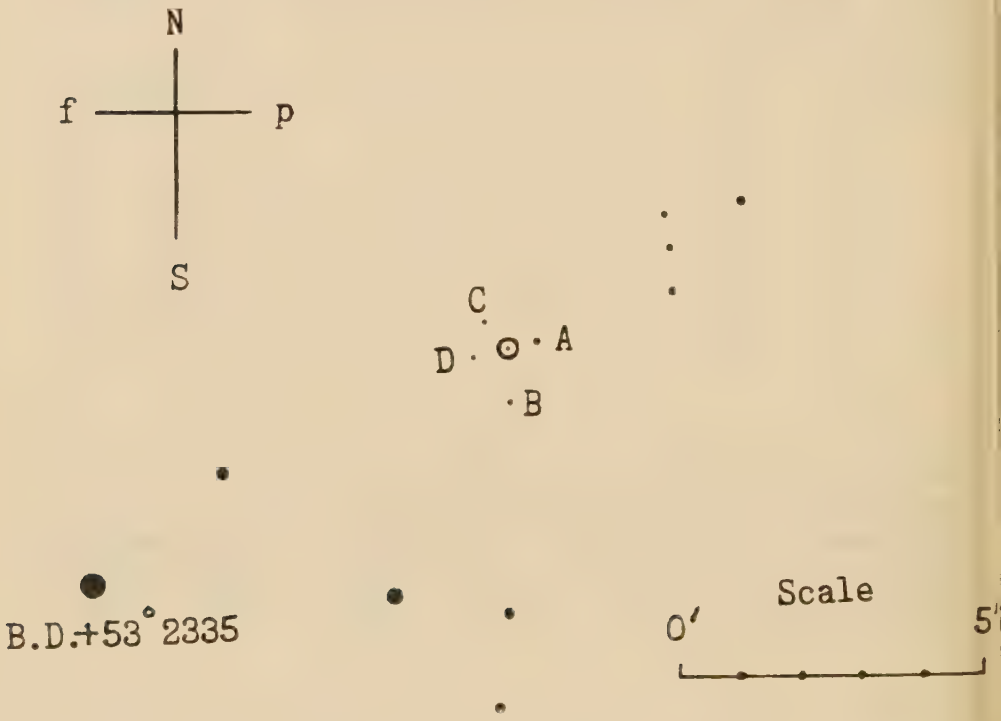


Fig. 1.

wrongly centred on the bright neighbour. The stars used are approximately at p.a. 302°, $\Delta = 2'{\cdot}0$, mag. 14·4, and at p.a. 243°, $\Delta = 1'{\cdot}1$, mag. 15·0.

Date	Mag.
1931 Oct. 5	14·7
Dec. 7	14·4
1932 Jan. 13	14·2

Nova Lacertæ (1910)

Observed on two nights. There has been no marked change in the brightness of this nova for many years, and it appears perfectly stellar and free from nebulosity.

Date	Mag.
1931 June 12	14·3
Oct. 20	14·2

Nova Ophiuchi (1848)

Observed on eighteen nights. This object is quite definitely variable, the observed range in 1931 being $0^m.9$. The image is less stellar than those of surrounding stars, but there is no nebulosity outstanding from the nova.

Date	Mag.	Date	Mag.
1931 June 12	12·74	1931 July 9	12·70
14	12·80	11	12·30
17	12·70	12	12·20
19	13·10	16	12·60
20	12·70	18	12·70
21	12·90	Aug. 14	12·70
24	12·70	15	12·75
July 3	12·60	16	12·80
6	12·70	17	12·90

Nova Persei (1901)

Observed on sixty-four nights. Variability has been marked, as usual. It seems to be, on the whole, of an irregular type, but occasionally an almost regular periodicity will manifest itself and persist for some weeks or months. Thus in 1931–1932 there is more than a suggestion of a regular fluctuation in a period of about nineteen days, and another in about six months. This sort of thing has happened in previous years, but the lengths of the periods are never the same for two years together. The total range of brightness observed in 1931–1932 amounted to $1^m.14$, rather smaller than that generally found.

Date	Mag.	Date	Mag.	Date	Mag.
1931 Aug. 22	13·18	1931 Nov. 7	12·90	1932 Jan. 15	13·25
26	13·67	9	12·74	20	13·57
Sept. 20	13·83	11	12·74	21	13·36
29	12·96	12	12·82	29	13·25
Oct. 2	13·18	13	12·79	30	13·77
5	12·86	19	13·00	Feb. 3	13·67
7	13·40	20	12·90	6	13·67
8	13·32	24	12·84	18	13·25
12	13·62	27	12·95	29	13·88
13	13·47	Dec. 4	13·26	Mar. 1	13·36
14	13·47	7	13·47	2	13·67
18	13·47	8	13·05	3	13·36
20	13·28	17	12·90	6	13·36
21	13·32	26	13·36	9	13·42
23	12·86	28	13·31	12	13·57
24	13·18	29	13·36	13	13·57
25	13·28	30	13·36	15	13·36
Nov. 1	13·18	31	13·36	16	13·30
2	13·18	1932 Jan. 8	13·15	24	13·42
4	13·10	11	12·90	25	13·42
5	12·90	13	12·90	Apr. 11	12·95
				22	13·57

OBSERVATIONS OF OCCULTATIONS.

S. Plakidis, D.Sc.

The observations given below have been made with the Doridis refractor (Gautier, 400 mm.) of the National Observatory of Athens. The geographical co-ordinates of this instrument are as follows :—

$\phi = +37^\circ\ 58'\ 15''{\cdot}5$ $\qquad$ $\log \rho \sin \phi' = 9{\cdot}78669$
$\lambda = 1^h\ 34^m\ 51^s{\cdot}93$ E. $\qquad$ $\log \rho \cos \phi' = 9{\cdot}89726$

(D. Eginitis, *Annales de l'Observatoire National d'Athènes*, vols. 5 and 12).

A power of 155 has been used throughout, and only immersions at the dark limb of the lunar disc were observed by the eye and ear method.

No.	Star	Mag.	Date 1932	Local Sidereal Time	Remarks
				h m s	
1	η Tauri	2·9	Jan. 18	2 32 18·2	Through clouds.
2	B.D. +23° 549	8·5	18	3 6 1·7	Through clouds.
3	+23° 560	8·0	18	3 51 12·7	Through clouds.
4	+23° 561	7·5	18	3 54 12·2	Through clouds.
5	B.D. +14° 306	8·8	Feb. 12	7 10 17·4	Fine.
6	+14° 308	8·8	12	7 16 11·3	Fine.
7	+14° 310	9·0	12	7 37 26·4	Hazy.
8	+14° 312	8·7	12	7 11 5·4	Fine.
9	+14° 313	8·7	12	7 26 38·5	Hazy.
10	B.D. +28° 1255	9·5	Mar. 16	6 49 25·7	Fine.
11	+27° 1248	9·0	16	7 22 27·0	Fine.
12	+28° 1258	9·1	16	7 40 8·5	Fine.
13	+28° 1264	9·3	16	8 13 17·0	Fine.
14	+28° 1265	9·2	16	8 15 48·3	Fine.
15	B.D. +23° 529	9·2	Apr. 9	8 29 54·9	Fine.
16	+24° 562	7·5	9	8 30 43·6	Fine.
17	+23° 540	7·0	9	8 39 56·6	Fine.
18	+23° 549	8·5	9	9 1 22·1	Fine.
19	+23° 553	7·8	9	9 7 18·9	Fine.
20	Paris Ph. +24° $3^h\ 44^m$ No. 110	10·9	9	9 15 12·4	Fine.
21	B.D. +23° 560	8·0	9	9 27 40·9	Fine.
22	+23° 561	7·5	9	9 31 31·5	Fine.
23	B.D. +27° 795	9·3	11	8 38 8·4	Fine.
24	+28° 823	9·5	11	8 46 20·8	Fine.
25	+27° 798	8·0	11	8 53 22·6	Fine.
26	+28° 824	9·5	11	8 57 47·7	Fine.

No.	Star	Mag.	Date 1932	Local Sidereal Time	Remarks
				h m s	
27	B.D. +28° 826	9·5	Apr. 11	9 13 6·6	Fine.
28	+28° 830	9·5	11	9 40 52·4	Fine.
29	+28° 831	9·1	11	9 45 24·4	Fine.
30	B.D. +26° 1572	9·3	13	9 53 52·6	Fine.
31	+26° 1574	9·3	13	9 58 41·9	Fine.
32	Paris Ph. +24° 8^h 24^m No. 115	9·8	14	9 45 4·7	Fine.
33	B.D. +23° 1954	9·4	14	10 1 15·8	Fine.
34	+23° 1955	9·2	14	10 2 13·7	Fine.
35	+23° 1960	7·8	14	10 16 45·5	Fine.
36	Paris Ph. +24° 8^h 24^m No. 130	9·2	14	10 17 8·5	Fine.
37	B.D. +24° 1958	8·7	14	10 24 6·5	Fine.
38	B.D. +19° 2201	7·5	15	9 58 37·6	Fine.